FINAL REPORT OF THE NATIONAL COMMITTEE OF FIFTEEN ON GEOMETRY SYLLABUS.

PRELIMINARY STATEMENT.

At the meeting of the National Education Association in Cleveland in 1908, the mathematics Round Table of the Secondary Department, numbering some two hundred members, unanimously called for a national committee to study and report upon the question of a syllabus for geometry. In December, 1908, the American Federation of Teachers of the Mathematical and National Sciences at its meeting in Baltimore authorized the appointment of a national committee of fifteen on geometry syllabus. At the Denver meeting of the National Education Association in 1909, the secondary department authorized the committee which had already been appointed by the American Federation to proceed under the joint auspices of the two national bodies.

The committee thus constituted was made up of eight representatives of secondary schools and seven representatives of universities, as follows: William Betz, East High School, Rochester, N. Y.; Edward L. Brown, North High School, Denver, Colo.; William Fuller, Mechanic Arts High School, Boston, Mass.; Walter W. Hart,* University of Wisconsin; Frederick E. Newton, Andover Academy, Andover, Mass.; Eugene R. Smith, The Park School, Baltimore, Md.; Robert L. Short, Technical High School, Cleveland, O.; Mabel Sykes, Bowen High School, Chicago, Ill.; Charles L. Bouton, Harvard University; Florian Cajori, Colorado College; Herbert E. Hawkes, Columbia University; Earle R. Hedrick, University of Missouri; Henry L. Rietz, University of Illinois; David Eugene Smith, Teachers College, Columbia University; Herbert E. Slaught, *Chairman,* University of Chicago.

* At that time head of the department of mathematics in the Shortridge High School, Indianapolis, Ind.

After an extended preliminary correspondence, the investigations of the committee were conducted in three subdivisions of five members each, the first dealing with "logical considerations," under the chairmanship of David Eugene Smith; the second having charge of "exercises and problems," under the chairmanship of Henry L. Rietz; and the third determining the "lists of theorems," under the chairmanship of Earle R. Hedrick.

These subcommittees carried on their work by correspondence and in some cases by meetings of their members during a period of a year and a half, the results being submitted from time to time to the general committee for suggestions and criticisms. A meeting was then held in Cleveland, Ohio, on November 24, 25, 26, 1910, at which were present the three subchairmen, the general chairman, and three other members. All the recommendations of the subcommittees were submitted to a searching examination in the course of which they were either passed, eliminated, amended, or reconstructed until finally substantial agreement was reached on all points.

These conclusions were then submitted to the remaining members of the committee for their approval and the report as thus evolved was put into type and a preliminary edition of two hundred copies was placed in the hands of a carefully selected list of critics for further suggestions. In this way the report finally reached the provisional form in which it was presented at the San Francisco meeting of the National Education Association in 1911, where it was discussed at great length and adopted with the understanding that the committee would again work it over in the light of all criticisms and suggestions and present it in final form at the next meeting. These instructions were followed with great care, and when all modifications had been made, upon which the members of the committee could reach substantial agreement, the report was again printed and distributed directly to two thousand members of mathematics associations throughout the country and indirectly, through the Bureau of Education at Washington, to some three thousand people upon personal request. In this final form it was presented at the Chicago meeting of the National Education Asso-

ciation in 1912, where it was unanimously adopted, and an appropriation was made for its further distribution.

In the preparation of this report the various syllabi published in this country and in foreign countries have been consulted. With one of these, the syllabus published by the Association of Mathematics Teachers in New England, this report makes direct comparison, indicating both omissions from, and additions to, that document. The historical introduction was prepared by Florian Cajori.

SECTION A. HISTORICAL INTRODUCTION.

Attempts Made during the Eighteenth and Nineteenth Centuries to Reform the Teaching of Geometry.

Bibliography.

The following six books bearing on the history of the teaching of geometry have been found most useful in making this compilation:

1. V. Bobynin—"Elementare Geometrie," being Chapter XXII. in Cantor's "Vorlesungen über Geschichte der Mathematik," Vol. IV., Leipzig, 1908, pp. 321–402. (Covers the second half of the eighteenth century.) Referred to as "Bobynin."
2. F. Klein—"Elementarmathematik vom höheren Standpunkte aus. Theil II.; Geometrie." Leipzig, 1909, pp. 433–515. Referred to as "Klein."
3. J. Perry—"Discussion on the Teaching of Mathematics." British Association Meeting, 1901. Referred to as "Perry."
4. H. Schotten—"Inhalt und Methode des planimetrischen Unterrichts." Leipzig, 1890. Referred to as "Schotten."
5. M. Simon—"Ueber die Entwicklung der Elementar-Geometrie im XIX. Jahrhundert." Leipzig, 1906. Referred to as "Simon."
6. A. W. Stamper—"A History of the Teaching of Elementary Geometry," New York, 1906. Referred to as "Stamper."

Other useful sources of information on the history of the teaching of geometry are as follows:

1. *Reports of the Association for the Improvement of Geometrical Teaching* (in England). The name of the association has been changed to "The Mathematical Association," and its present organ is the *Mathematical Gazette.*

2. *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht,* Leipzig and Berlin. Formerly called "Hoffmann's Zeitschrift," now "Schotten's Zeitschrift."
3. *L'Enseignement Mathématique. Revue internationale.* Paris.
4. *La Revue de l'Enseignement des Sciences.* Published monthly. Paris.
5. *Nature* (London). See Indexes for "Geometry."
6. D. E. Smith—"The Teaching of Geometry." Boston, 1911.
7. G. Loria—"Vergangene und künftige Lehrpläne." Deutsch von H. Wieleitner, Leipzig, 1906.
8. G. Loria—"Della varia fortuna di Euclide." Rome, 1893.
9. R. Fricke—"Ueber Reorganisationsbestrebungen des mathematischen Elementarunterrichts in England." *Jahresbericht d. deutsch. Math. Vereinigung,* Vol. XIII., 1904, p. 283, etc.
10. Klein-Schimmack—"Vorträge über den mathematischen Unterricht an den höheren Schulen." Leipzig, 1907.
11. *Plan d' études et programmes d' enseignement dans les lycées et collèges de garcons.* Paris, 1903.
12. A. W. Stamper's list of references at the end of his book.
13. The various histories of mathematics.

France.

France began to maintain a critical attitude toward Euclid as a text-book in geometry for beginners as early as the time of Petrus Ramus (1580). Ramus treated geometry as the art of accurate measurement. In the eighteenth century this spirit of independence was intensified by the publication of Clairaut's "Eléments de Géométrie" (1741), in which surveying and other practical matters received marked attention. In the latter half of the eighteenth century Euclid ceased to be used as a text-book in France.

Williamson, in his edition of Euclid, 1781, criticises Clairaut as follows: "Elements of geometry carefully weeded of every proposition tending to demonstrate another; all lying so handy that you may pick and choose without ceremony. 'This is useful in fortification;' 'you cannot play at billiards without this.' 'You only look through a telescope like a Hottentot until this proposition is read,' with many such powerful strokes of rhetoric to the same purpose. And upon such terms, and with such inducements, who would not be a mathematician? Who would go to work with all that apparatus which I have described as necessary for understanding Euclid, when he has only to take a pleasant walk with Clairaut upon the flowery banks of some

delightful river, and there see, with his own eyes, that he must learn to draw a perpendicular before he can tell how broad it is?" About 1836 De Morgan remarks that these arraignments are not "without their force, when directed against experimental geometry as an ultimate course of study, [but] lose their ironical character and become serious earnest, when applied to the same as a preparatory method." De Morgan strongly favors a geometry like Clairaut's as a preparatory course.

The critical attitude of Ramus and Clairaut toward the "Elements" of Euclid brought to the mind of D'Alembert the questions: "What are the elements of a science? What should be the contents of a book called elements?" D'Alembert gives his answers in two articles, "Eléments des sciences," and "Des éléments de géométrie" in the Encyclopédie méthodique (about 1784).*

D'Alembert distinguishes between two kinds of elements of a science:

(1) If all truths or theorems of a science which are the foundation for all others are brought together, so that these truths or theorems potentially comprise the whole science, then these constitute, when properly coördinated, the *elements* of the science. In geometry, such elements embrace not merely the principles of mensuration and the properties of plane figures, but also the application of algebra to geometry, and the differential and integral calculus in its application to curved lines.

(2) The elements of a science may be defined also as comprising those truths or theorems which treat the subject matter in the simplest way, and which constitute, together with their deductions, a detailed study of the simplest parts of the science. By the elements of geometry, elements of this kind are usually meant; they include only the properties of plane figures and the circle.

Dissatisfied with the elements of geometry known in his day, D'Alembert sets up the following demands which such texts should fulfill:

* Encyclopédie methodique, Mathématiques I., 617–625, III., 133–136. We have used the Italian translation of this dictionary, Padova, 1800, and also a full abstract of these articles, given by Bobynin. See Bobynin, p. 325, etc.

(1) The text should develop the subject along the path pursued by the discoverers of the science, so as to show the truths in their natural relations to each other.

(2) The usual division of the subject into longimetry, planimetry, and stereometry does not provide for the circle and sphere and is therefore inadequate. The division into plane geometry and solid geometry, D'Alembert does not consider at all. He suggests the division into the geometry of the straight lines (considered with respect to position and relative magnitude) and circles, the geometry of surfaces and the geometry of solids. The straight line and circle must be taken up together. The circle renders immense service in considering the position of lines. The measurement of angles by circular arcs and the principle of congruence constitute the basis of the first part of the geometry of lines, upon which other theorems of this part rest. The second part in the geometry of the straight line has as its fundamental theorem the one on the section into proportional parts of two sides of a triangle by a line parallel to the third side. This involves incommensurables.

(3) Incommensurable relations must be treated by the apagogic method, according to which it is shown that one ratio cannot be greater or smaller than a certain other ratio, hence it must be equal to that other ratio. He uses this for the following reasons: Incommensurable magnitudes involve the idea of the infinite and therefore, he claims, cannot be treated by any direct method. Notwithstanding this difficulty presented by incommensurable lines, he maintains that they should be taken up early in geometry, because of their importance. He states that the whole theory of incommensurables demands only one theorem, concerning the limits of quantities, viz.: "Magnitudes which are the limits of one and the same magnitude, or magnitudes which have one and the same limit, are equal to each other." In the geometry of the circle, of surfaces and solids, he feels that the method of exhaustion or that of limits should be used.

(4) A suitable text on the elements of geometry can be prepared only by a mathematician of the first rank. D'Alembert complains that most elementary geometries are written by men of little ability.

(5) To lay down definitions at the beginning without any analysis of the subject is not only contrary to sound philosophy but contrary to the natural march of thought.* Axioms are useless.

Ideas similar to those of D'Alembert are embodied in a text on geometry by Louis Bertrand of Geneva,† who in Berlin had been close to Euler. Bertrand's book antedated D'Alembert's articles in the Encyclopédie Méthodique. Like D'Alembert he divides geometry into three parts: (1) Geometry of line and circle, (2) Measurement of parts of a plane bounded by straight lines and circles, (3) Measurement of curved surfaces and solids. Bertrand ignored the classification of geometry into plane and solid. His second theorem is: "When two planes intersect, their common section is a right line." The straight line and circle are taken up together at the beginning as D'Alembert would have it. The incommensurable case is treated by the *reductio ad absurdum* method. In the latter part of the geometry he uses also the method of exhaustion. Bertrand reduces the number of theorems, in one instance, by replacing theorems on the mensuration of prisms, pyramids, cylinders, cones, and spheres with the corresponding problems.

Bertrand's work was published in two unwieldy volumes and had little sale, yet exercised some influence, particularly upon Lacroix, whose "Cours de Mathématiques," published at the close of the eighteenth century, has been used until recently. Lacroix divides his geometry into geometry of the plane and geometry of space, and does not follow D'Alembert closely. According to Lacroix there are only two kinds of theorems that should find a place in an elementary geometry: (1) Theorems necessary for the comprehension of the line of argument, developed synthetically. (2) Theorems which grow out of the practical operations in geometry (drawing and measuring). He objects to placing all axioms at the beginning, believes in the omission of the definition of an angle, favors "a straight line is the shortest path between two points" as growing out of the

* See "Axiome" and "Courbe" in Encycl. Méth.

† Développment nouveau de la partie élémentaire des mathématiques, Geneva, 1778.

child's experience, and uses the apagogic method for incommensurables.

Another author of note was Bézout, who followed D'Alembert's plan quite closely, but was criticized for his lack of rigor and for his endeavor to lighten the work of the examiner as well as of those being examined.*

The most celebrated work on elementary geometry is that of Legendre (1794). He came nearest to fulfilling D'Alembert's requirement that the elements be written by a mathematician of the first rank. He does not follow D'Alembert's plan for a book on geometry, nor does he heed the philosophic demand that the author should follow the path of the originators of the science. Impressed by the lack of rigor in the works of his day, he aims at greater rigor and approaches closer to Euclid than his predecessors had done. He does not divide geometry in the manner of D'Alembert and Bertrand. Like Euclid, Legendre begins with definitions and axioms. The first four chapters are given to plane geometry, the last four to solid. The first book treats of the equality of angles and triangles, the second of the circle and the measurement of angles, the third of proportional figures, the fourth of regular polygons and the measurement of the circle. Legendre uses in measurement the terms *equal* and *equivalent*. He uses the *reductio ad absurdum* method for incommensurables and the method of exhaustion for curved lines.

What was it that made this book so successful? In the first place must be mentioned his great clearness of exposition and his attractive style. A great advance of Legendre over Euclid was the fuller treatment of solid geometry. He leans less toward logic and more toward intuition than does Euclid. In place of Euclid's famous fifth book on incommensurables, Legendre borrows rational and irrational numbers from arithmetic, even though in arithmetics no scientific treatment of those subjects was given in his day. A theorem true for rationals is assumed to be true for irrationals. Thus, if $A:B=C:D$, then $AD=BC$ in all cases. Klein says that this is in accordance with the practice of the best mathematicians of his day, that even Lagrange works out the expansion of $(x+h)^n$ when n is

* Bobynin, p. 355.

rational and assumes the results thus obtained to be true for irrational values of *n*. Legendre stands for a fusion of geometry, not only with arithmetic, but also with trigonometry. As late as 1845 Legendre's geometry still contained trigonometry, but as Klein remarks,* the trigonometry and the practical applications of geometry were gradually filtered out. Comparing A. Blanchet's edition of 1876 with an edition of 1817, we find also that the twelve "notes" on topics of elementary geometry, covering 55 pages in the older edition, are omitted in the later edition. The later edition has a somewhat fuller treatment of solid geometry and a list of exercises in original proofs, loci, and constructions. Other notable changes were made in the 1845 edition by J. B. Balleroy and A. L. Marchand. They state that Legendre uses the *reductio ad absurdum* method to excess, a method which "convinces but does not satisfy the mind." Legendre's text is, however, left intact, alternative proofs being given in notes at the end. These alternative proofs, as well as the proofs given in the modified text of the 1876 edition, are rough applications of the theory of limits.

During the first half of the nineteenth century, and even later, the works of Legendre, Lacroix, and Bézout were used extensively in France. In later editions less stress was laid upon practical applications and numerical computation. Otherwise few changes occurred. In general, school organization, based on the regulations of the time of Napoleon I., was quite fixed in France until 1870. France has a rigid centralization of authority in education. If the "Conseil d' instruction supérieure" decides upon a change, the whole country adopts it at once. As compared with the German, the French teacher has little individual freedom. France is a country with a "system of revolutions from above."† Since 1870 the movement has been toward greater individual freedom. The later tendencies in geometry are imaged in the work of Rouché and de Comberousse, which contains a large amount of new material and meets the demands of the one year course of the *classe de mathématiques spéciales* during which as much as seventeen hours

* Klein, p. 470.

† Klein, p. 457.

per week are given to mathematics and a degree of specialization is allowed in preparation for university courses, as in no other country. In 1902 and 1905 new official courses of study were adopted in France in which greater stress is laid upon graphic representation, the idea of a variable and a function, and upon the practical applications of mathematics. The introduction of the derivative in algebra in the regular secondary classes, the use of motion in geometry, and emphasis on geometric drawing are other characteristic features of this reform. This new tendency is mirrored in the geometry of E. Borel, a valuable book, in which the practical receives due emphasis and in which intuition meets with fuller recognition. With Borel the concept of motion is prominently used. There is an introduction of eight pages on the use of the ruler, compasses, and protractor, and ten pages on the mensuration of surfaces and solids, treated empirically. Applications are skillfully interwoven with theory throughout the book. He has well-selected practical exercises involving symmetry, the nets of regular polygons, the use of pulleys, and so on. Algebraic geometry and the development of metric properties come last in the book. He introduces the rudiments of trigonometry. The usual division into plane geometry and solid geometry is not rigidly maintained. A similar tendency is shown in the recent geometries by C. Bourlet (1908), Fort and Dreyfus (1908), and Niewenglowski (1910).

A parallel and somewhat different tendency in France is seen in the geometry of Ch. Méray of Dijon, which was first brought out in 1874 but has only in recent years received much attention. Méray represents the severely logical mode of exposition;* he uses in his proofs no fact of observation which has not been previously set down in an axiom; he formulates a complete list of axioms, but introduces each only when it is needed; nor does he aim to limit their number to a minimum. Characteristic of Méray is the complete fusion of plane and solid geometry, and the use of motion, not only as a means of proof, but also to define parallels. Recently there has been considerable discussion in France on the question whether in laying the foundations to geometry, *motion* should be used or not. The defenders

* Klein, p. 475.

of a static theory of parallels claim that motion cannot be visualized on the board, rendering intuition more difficult. The defenders of the kinematic theory advocate the use of movable figures.* Prepared under the influence of Méray, so far as the use of motion is concerned, are the geometries by Borel, Bourlet, Fort and Dreyfus, and Niewenglowski, above mentioned.

Influenced by the Perry movement in England and America, France is experimenting on the laboratory method of instruction.† A mathematical laboratory has been recently founded by J. Tannery and E. Borel at the Ecole Normale supérieure in Paris in order to give prospective instructors an insight into the possibilities of this method of teaching.

Germany.

Klein‡ expresses surprise that, during the Renaissance, Euclid should have come to be looked upon as a text suitable for the first introduction in geometry. Perhaps the reason for this attitude toward Euclid lies in the fact that geometry was first taken up in the universities by students of maturer years. As geometry came gradually to be taught to younger and younger pupils, Euclid was still retained. Thus the misconception arose that Euclid was a suitable geometrical text for young boys.

While D'Alembert formulated his ideas on elementary geometry in France, A. G. Kästner evolved in Germany a type of his own, in his work, "Anfangsgründe der Arithmetik, Geometrie, Trigonometrie und Perspectiv," Göttingen, 1758. Kästner begins with definitions and axioms in Euclidean style, develops the geometry of the plane (69 pages) and ends this part with practical applications (47 pages). The second part of the geometry begins with the geometry of space (60 pages), continues with 31 pages given to plane trigonometry and its applications to the solution of triangles, and with 9 pages of practical geometry. Then follow spherical trigonometry and 24 pages on perspective. The method of exhaustion is used. It

* Schotten, *Zeitschrift,* Vol. 40, 1909, p. 445; *La Revue de l'Enseignement des Sciences,* many articles in Vols. 1–3.

† Schotten, *Zeitschrift,* Vol. 40, 1909, pp. 444–5; *L'Enseignement Mathématique,* 11, p. 206.

‡ Klein, p. 434, 435.

was the opinion of Kästner that "the newer works on geometry lose the more in clearness and thoroughness, the farther they depart from Euclid." He complains that modern authors, particularly the French, have departed from the ancient rigor, "to make the study of mathematics easier for people whose main occupation is not study, namely for soldiers."

Not without interest is W. J. G. Karsten's "Lehrbegriff der gesamten Mathematik," in eight volumes, 1767–77, the first two volumes of which are given to arithmetic and geometry. Karsten begins with arithmetic, then proceeds to plane geometry, closing with simple arithmetical applications. He proceeds thereupon to solid geometry, returns to arithmetic, and gives the rudiments of algebra with logarithms, followed by trigonometry and its applications to plane geometry. Finally are given the rudiments of spherical trigonometry and a fuller treatment of solids. Nowhere are heavy demands made upon the pupil. That this exposition was intended for students of university grade, rather than those in the preparatory school, testifies to the low state of mathematical instruction in German universities of the eighteenth century. Close relation between arithmetic, geometry, and trigonometry is also maintained in the works of J. G. Büsch (1776) and G. S. Klügel (1798), the aim being to make the subject easy of comprehension.

In the nineteenth century, until near its end, advanced mathematicians in Germany took little or no part in the improvement of the teaching of elementary mathematics. In geometry, Euclid's text was not usually taught, but the dogmatic method of Euclid was in vogue during the first half. About the middle of the century Euclid's order of the theorems came to be criticized as chaotic. It is interesting to see the Germans attack Euclid's order as arbitrary and the English defend it as the only order worthy of serious consideration. The grouping of theorems according to subjects came to be discussed in Germany.* The advocacy of object teaching by Pestalozzi, the championing of Pestalozzianism by Herbart, the attacks upon mathematical reasoning and particularly upon Euclid that were made by Schopenhauer† conspired to influence the teaching of geometry.

* Schotten, p. 11.

† Klein, p. 503.

About 1860 the genetic method (called "heuristic" when the inventional side was emphasized) came to be discussed. This makes a plea of being a natural method, since it incites self-activity in the pupil. With the genesis of a theorem the pupil sees intuitively its inner relation to other theorems; he not only sees whence he came but also whither he is going; the reader of Euclid is blindfolded, so to speak, and then somehow transported to the next station. It is difficult to prepare text-books for the genetic method. The teacher by careful questioning one moment leads the student, the next moment follows him, and no one can foresee the exact path which this mode of advance will mark out. It is not strange, therefore, if many teachers proceeded heuristically while the texts retained mostly the dogmatic form.* Moreover, experience made it plain to teachers that the dogmatic statement of theorems has a high mnemotechnic value.† While the genetic method in its pure form has not succeeded in establishing itself, it has exerted a strong influence by shifting the emphasis from the memorizing of proofs to the cultivation of originality and logical reasoning.

Another movement that sprang from the teachings of Pestalozzi and Herbart was the adoption of preliminary courses on observational geometry and drawing, about 1870. Such courses had been recommended long before this time. This movement was stronger in Germany than in England and France. In their propædeutic courses the geometry of solids was to receive consideration and a taste of the genetic method was recommended. The pupils acquired dexterity in the use of ruler and compasses. Propædeutic courses have maintained their place to the present time.

Herbart made strong endeavors to remove the superstition that had arisen in early days when Euclid was placed in the hands of young and immature students, to the effect that mathematics could be learned only by a few pupils endowed with special gifts. According to his view the fault lies as a rule in the abstract character of the early instruction; the introduction of propædeutic courses and the greater emphasis upon "Anschauung" at all stages had shown that most students can

* Schotten, p. 96.

† Schotten, p. 13.

master mathematics. Whether "amathematicians" do exist in rare instances, is a question which Klein refers to experimental psychologists for reply.*

A third movement, agitated in Germany, was in favor of the introduction into elementary instruction of the concepts of the modern projective geometry. It originated about 1870.† The criticism was made that Steiner, Möbius, and von Staudt had been so busy with their researches as to make no attempt to reform elementary instruction, and that text-book writers had ignored the researches of these great men. The leaders in this attempt to incorporate modern methods were Schlegel and Fiedler. A concomitant of this programme was the breaking down of the division of geometry into plane and solid, and the effort by the use of models, etc., to make geometry more concrete. To effect this reform, a number of texts by Schlegel, Müller, Kruse, Becker, Worpitzky, Henrici, and Treutlein sprang into existence.‡ Aside from the production of interesting text-books this agitation has had little success. The books in question were seldom used.§ Can it be that D'Alembert's dogma is, after all, based upon truth—the dogma that the historical order of development of geometry is the pedagogical order; that is, the easiest approach to the science for the young mind? Are the concepts of projective geometry more difficult to grasp than those of the older geometry, or did the texts just named overtax the pupils, and perhaps in other ways violate the demands of sound pedagogy?

Most interesting are the statistics gathered in Prussia in 1880 which showed the following distribution of geometrical texts: Kambly was used in 217 institutions; Koppe in 54; Mehler in 44; Reidt in 29, while 55 texts were used in one institution each. Kambly's "clever but unscientific book" was first issued in Breslau in 1850 and a few years ago reached the 101st edition in the revision by Roeder. Koppe was looked upon as an inferior work, yet it enjoyed great popularity. On the other hand, books like those of H. Müller and even Henrici and

* Klein, p. 499.

† Schotten, p. 18.

‡ Schotten, p. 19.

§ Schotten, p. 20.

Treutlein seldom passed beyond the second edition. This most astonishing success of works considered as scientifically inferior requires explanation. Schlegel says* "that the quality of the books most widely adopted allows one to draw an inference respecting the scientific level of the instruction generally reached in that subject." But this remark considers merely one phase of this question. May not the mass of teachers have had a feeling or insight concerning text-books which involved questions of intuition or other psychologic matters that the writers of the more scientific books overlooked? Simon† points out that until recently the German teacher, unlike the French, enjoyed complete freedom in teaching, and that small texts, like Kambly, allow his individuality much wider play.

Kambly's "Elementar-Mathematik" was made up of four parts: first, arithmetic and algebra; second, planimetry; third, plane and spherical trigonometry; fourth, stereometry. Of interest here is the interpolation of trigonometry. We have before us Kambly's *Planimetrie,* 43d edition, Breslau, 1876. Among the points of popularity we mention the following:

1. The book contains only as much matter as a class can conveniently finish in one year. Skipping parts of a book, says Kambly, has a bad effect upon both pupils and parents.

2. The diction is clear and simple. Mathematical symbols are used freely. The setting of the type is such as to enable the eye more quickly to see the relations set forth.

3. The arrangement of the book is such as to allow the teacher much freedom. He may, for instance, omit incommensurables altogether, or else substitute for certain proofs in the regular text others given in the footnotes where rough proofs are found for incommensurable cases.

4. Easy arithmetical applications, original theorems, and original constructions are given at the end of the book, so that some, or all, may be conveniently taken or omitted, according to the preference of the teacher.

Koppe's "Planimetrie" made somewhat greater demands upon the powers of the pupil than did Kambly, but incommensurables were treated only in footnotes or in remarks fol-

* Schotten, p. 21.

† Simon, p. 25.

lowing the proofs of theroems. In Lübsen's "Elementar-Geometrie," I have not been able to find a reference to incommensurables. It differs from Kambly and Koppe in having better figures and in having them on the page where they are needed, instead of at the end of the book on separate sheets that unfold. A clever feature in Lübsen are the practical applications introduced from the very beginning. How to run a straight line over undulating country by the use of poles, is explained in several diagrams on the first pages. Other figures show how to determine the distance between points on opposite banks of a river.

Since about 1890 the activity of Felix Klein of Göttingen, in mathematical reform, has been very great. For the first time since the death of Kästner, is the influence of university professors upon the teaching of elementary mathematics in Germany beginning to be strongly felt. Among the defects of geometrical instruction, he points out the insufficient fusion of the various branches of elementary mathematics.* Thus, too little attention is given to drawing of solids and to projection, to the idea of motion in a figure to replace Euclidean rigidity, to the fusion of arithmetic and geometry, to the introduction of the coördinate representation of analytics. On the other hand, the construction of triangles from given data is over emphasized,† as is also the study of the curious points and lines in the geometry of the triangle. This last criticism applies even more strongly to English text-books.

Klein points out that modern demands in geometric teaching, *first,* emphasize the psychologic point of view,‡ which considers not only the subject matter, but also the pupil, and insists upon a very concrete presentation in the first stages of instruction, followed by a gradual introduction of the logical element; *second,* call for a better selection of the material from the viewpoint of instruction as a whole; *third,* insist on a closer alignment with practical applications; *fourth,* encourage the fusion of plane and solid geometry, and of arithmetic and geometry.§

* Klein, p. 439.
† Klein, p. 442.
‡ Klein, p. 435.
§ Klein, p. 437.

The "Lehrbuch der Mathematik nach modernen Grundsätzen" by Behrendsen and Götting, a secondary school text published in 1909, carries out Klein's ideas on the teaching of algebra and geometry.

A piece of research of vital importance in the advanced study of geometry is the "Foundations of Geometry," brought out in 1899 by Professor Hilbert of Göttingen.* Though widely read by mathematicians, it has exerted no direct influence upon elementary teaching in Germany. It has been felt that this mode of treatment is not suitable for pupils first entering upon demonstrative geometry.

ITALY.

Since the unification of Italy, great mathematical activity has existed in that country. Before that event, very different practices in geometrical teaching existed in different parts of the country.† In 1868 Cremona and Battaglini were members of a government commission to inquire into the state of geometrical teaching. They found it unsatisfactory, and the number of bad text-books so great, and so much on the increase, that they recommended for classical schools the adoption of Euclid, an edition of which was brought out by Betti and Brioschi. Later other works of scientific merit replaced Euclid. Cremona's great emphasis upon projective geometry reached from the universities down into secondary schools. A typical work is that of A. Sannia and E. d'Ovidio, 1869, which uses the theory of limits and retains the division of geometry into plane and solid. It stands closer to Euclid than to Legendre. The blending of plane and solid geometry, which received great emphasis in Italy, is typified in the "Elementi di Geometria" of R. de Paolis, 1884.

A very remarkable school came into being in Italy, the purpose of which is to render geometry still more rigorous than in the Euclidean text. Starting with a single basic concept, the point, all other concepts are to be logically developed. This movement is typified in the works of G. Veronese.‡ Of ele-

* D. Hilbert, "Grundlagen der Geometrie," in *Festschrift zur Feier der Enthüllung des Gauss-Weber-Denkmals in Göttingen*, Leipzig, 1899.

† Simon, p. 43.

‡ Klein, p. 482.

mentary works he has prepared "Nozioni Elementari di Geometria Intuitiva," 1902, and "Elementi di Geometria," 1904, the first of these being a propædeutic work. Demonstrative geometry is taken up in Italy with older pupils than in Germany and the United States; hence works of greater rigor can be used. Veronese endeavors to state all the necessary postulates of geometry, no matter how obvious, as for instance, "There exist *different* points," to make it plain that we do not consider a geometry in which only one point exists.* As regards the selection of material, Veronese confines himself mainly to that of Euclid, thus receding from the tendency of the School of Cremona. He avoids all fusion with arithmetic. Somewhat similar in character is the "Elementi di Geometria" of F. Enriques and U. Amaldi, 1905.

The effort at rigor, due to Veronese, has been intensified in the great school of Peano, which endeavors to eliminate all intuition. It seems that this school has influenced even elementary instruction and the teaching in technical schools.† This recent Italian emphasis upon extreme rigor has led to deplorable results with the less gifted pupils, and a reaction appears to be setting in. Under the leadership of Loria and Vailati a movement is on foot favoring greater emphasis upon intuition, the introduction of some modern geometrical notions, the fusion of geometry with arithmetic, and the concession to the demands for practical applications made by this age of industrial development. In fact, Italy is entering upon a reform much like that of Germany and France.‡

ENGLAND.

Roger Bacon says that toward the close of the thirteenth century the definitions and a few of the theorems in geometry were studied by some pupils at Oxford.§ About 1570 Sir Henry Savile began to lecture at Oxford on Greek geometry, and in 1619 Briggs at Cambridge on Euclid. In 1665, Isaac

* Klein, p. 483.

† Klein, p. 486.

‡ For additional details see W. Lietzmann's article in Schotten's *Zeitschrift,* Vol. 39, pp. 177–191; Vol. 40, pp. 227–228.

§ Ball, "Mathematics at Cambridge," 1889, p. 3.

Barrow at Cambridge prepared a complete edition of Euclid, which was the standard for fifty years. Gow says that "the seventy years or so, from 1660 to 1730, when Wallis and Halley were professors at Oxford, Barrow and Newton at Cambridge, were the period during which the study of Greek geometry was at its height in England."* In 1703, William Whiston became the successor of Newton at Cambridge. He brought out an edition of Tacquet's Euclid. Robert Simson's edition of Euclid first appeared in 1756. Simson was professor of mathematics at the University of Glasgow. In the universities of Great Britain, Euclid met with no competition. Ward's "Young Mathematician's Guide," 1707, may have been used to some extent, but probably more for its arithmetic and algebra than for its geometry. Practical men, holding positions as excise officers, had to be familiar with practical geometry. For them practical treatises existed, some of which gave explanations of the slide rule. A departure from Euclidean rigor might be expected in the education of men for the army or navy. We have seen that Kästner criticized the French for making mathematics easy for men interested in war. England has had since 1722 an academy at Portsmouth where men spent one or two years studying navigation, drawing, etc. England has had also, since 1741, a military academy at Woolwich, where sons of noblemen and military officers were taught fortification, gunnery and mathematics. Among the mathematical professors at Woolwich, during the eighteenth century, were Thomas Simpson, John Bonnycastle and Charles Hutton, all three authors of text-books, including geometries. Hutton's works went through several editions in the first half of the nineteenth century. From this it is evident that Euclid did not hold universal sway in England. Yet the forces opposing him were utterly unable to dislodge him.

In 1822, Sir David Brewster brought out an English translation of Legendre's geometry. Did teachers rally in favor of the introduction of this text? We shall see that DeMorgan suggested the use of some parts of it on solid geometry; DeMorgan deplored that solid geometry was seldom or never taught before trigonometry. But otherwise we are not able to find any serious reference to this translation of Legendre.

* Gow, "History of Greek Geometry," Cambridge, 1884, p. 208.

During the second half of the eighteenth century England had come to be the only country where Euclid was practically the only geometrical text used. During the eighteenth century the average age of freshmen in the English universities was gradually increasing, and perhaps at this time, Euclid passed from the universities to the lower schools. There is no explicit proof, however, that in the great "public schools" Euclid was studied before the nineteenth century.*

Very recently† some interesting information has been published about one of the "public schools"—Christ Hospital—which paid more than usual attention to mathematics in the courses for boys preparing to enter the royal navy. It seems that as early as 1680 such boys were required to study the earliest parts of the first book of Euclid, the 10th, 11th, and 12th propositions of the sixth book, and to learn arithmetic. Perhaps this represented all the theoretical mathematics taught, for Sir Isaac Newton, whose advice about changes in the course was sought, notes the following omissions: There was no "symbolic arithmetic," no "taking of heights and distances and measuring of planes and solids," no "spherical trigonometry," nothing of "Mercator's chart." In other "public schools" probably no courses in geometry were given during the eighteenth century. Says Stamper: "It was not until about the middle of the nineteenth century that the study of Euclid became common in the secondary schools of England."

It would be instructive to secure more information explaining how it was possible for Euclid to maintain its supremacy as a text, when geometry was being transferred from the universities to the schools. What were the experiences of teachers in secondary schools with the Euclidean text? The desirability of modifying Euclid must have arisen early, for in 1795 John Playfair brought out a revised Euclid containing the first six books and adding the computation of π and a book on solid geometry drawn from modern sources. Playfair endeavored to give the geometry a form which would render it more useful. Euclid's fifth book, which had never been used successfully

* Stamper, p. 88.

† "A School Course in Mathematics in the XVII Century," by W. W. R. Ball, in the *Mathematical Gazette*, Vol. V., 1910, Part I., pp. 202–205.

with beginners in geometry, as far as we can ascertain, was modified by Playfair by replacing Euclid's prolix explanations by the more concise language of algebra. But Playfair did not try to change the nature of the reasoning. Had there been a strong movement against Euclid in England at this time, Playfair would probably have joined it. In his review of Leslie's "Geometry" in the *Edinburgh Review,* Vol. 20, 1812, p. 79, he says: "A question has been sometimes agitated whether it is most advantageous, for the study of geometry, to possess a number of elementary treatises, or to have one standard work, like that of Euclid . . . the same lessons are not suited to every intellect, and on these accounts it may be of advantage that different elementary texts should exist. We are very much inclined to the latter opinion."

William George Spencer's unique booklet on "Inventional Geometry" was brought out about 1830 or 1835, but "received but little notice" at that time. A noteworthy device for aiding the young mind through sensuous stimulus was the use of colored diagrams, suggested by Oliver Byrne, in his edition of Euclid, London, 1847. The failure of this book is doubtless due to the want of moderation in the use of colors.

The ablest writer on the teaching of elementary geometry during the first half of the nineteenth century in England was Augustus DeMorgan. His articles published in the *Quarterly Journal of Education,* in 1831, 1832, and 1833, display a pedagogical insight which would have prevented many calamities in English teaching, had his views been more promptly and widely accepted. Elsewhere we quoted DeMorgan's remarks on Williamson's criticism of Clairaut's geometry, which showed that DeMorgan firmly believed in a preliminary course in geometry, as an introduction to a logical course like that of Euclid. It will appear that England was the last country actually to introduce propædeutic courses in elementary instruction.

DeMorgan did not hesitate to recommend radical changes in Euclid. Here is what he said in 1831, in an article in the *Quarterly Journal of Education,* entitled "On Mathematical Instruction":

"With regard to the fifth book of the 'Elements,' we recommend the teacher to substitute for it by the common arithmetical

notions of proportion. Admitting that this is not so exact as the method of Euclid, still, a less rigorous but intelligible process is better than a perfect method which cannot be understood by the great majority of learners. The sixth book would thus become perfectly intelligible."

Two years later, in an article in the same journal "On the Methods of Teaching the Elements of Geometry," DeMorgan dares to suggest that certain parts of Legendre might be profitably substituted for parts of Euclid. "The eleventh book of Euclid may, in our opinion, be abandoned with advantage in favour of more modern works on solid geometry, particularly that of Legendre, which the English reader will find in Sir David Brewster's Translation." In the same article DeMorgan gives utterance to a difficulty experienced by young students, which has been referred to by many writers in different countries, the *reductio ad absurdum*. DeMorgan says: "The most serious embarrassment in the purely reasoning part is the *reductio ad absurdum*, or indirect demonstration. This form of argument is generally the last to be clearly understood, though it occurs almost on the threshold of the 'Elements.' We may find the key to the difficulty in the confined ideas which prevail on the modes of speech there employed." As regards the difficult fifth book, DeMorgan said, in 1833, "We would say to all, teach the fifth book, *if you can;* but we would have all remember that there is an *if*." In another place he adds: "We strongly suspect that Euclid, as studied, does as much harm as good." To the credit of teachers be it said, that the fifth book was quite generally omitted. But DeMorgan's activity in this line did not end here. In 1836 he published "The Connexion of Number and Magnitude; An attempt to explain the fifth book of Euclid." For fifty years this tract was not duly appreciated; later it began to wield a wide influence; it is on this tract that the substitute for the fifth book given in the Syllabus of the Association for the Improvement of Geometrical Teaching is modeled; it is on this tract that the revised fifth book in the more recent editions of Euclid by Nixon and by Hall and Stevens is based.

The need of modifying the text of Euclid is brought out by DeMorgan in the "Companion to the British Almanac" of

1849, page 20, as follows: "If the study of Euclid has been almost abandoned on the continent, and has declined in England, it is because his more ardent admirers have insisted on regarding the accidents of his position as laws of the science."

How little influence DeMorgan's views wielded in England before about 1870 as regards the revision of Euclid's fifth book and the study of solid geometry, appears from the fact that the most popular edition of Euclid for many years was the one brought out in 1862 by Todhunter. This author reproduces Simson's text, though he greatly assists the pupil in overcoming the difficulties by breaking up the demonstrations into their constituent parts. In an Appendix are given notes, supplementary propositions and original exercises. Todhunter was quite out of sympathy with the purposes of the Association for the Improvement of Geometrical Teaching.*

Opponents of Euclid existed in England at all times. Thus in 1860 W. D. Cooley brought out a rival text. Eleven years later he expressed himself regarding this venture as follows:†

"In 1860 there was published for me, by Messrs. Williams and Norgate, a little volume entitled, 'The Elements of Geometry Simplified and Explained,' adapted to the system of empirical proof, and of exhibiting the truth of theorems by means of figures cut in paper. It contains in 35 theorems the quintessence of Euclid's first six books, together with a supplement not in Euclid. There was no gap in the sequence or chain of reasoning, yet the 32nd and 47th propositions of Euclid were, respectively, the 3d and 17th of my series. This book proved a failure, for which several reasons might be given, but it will be sufficient here to state but one, namely, that it came forth ten years before its time."

The reformers found a champion in Sylvester, who in 1869, before Section A of the British Association, exclaimed: "I should rejoice to see mathematics taught with that life and animation which the presence and example of her young and buoyant sister (natural science) could not fail to impart, short roads preferred to long ones, Euclid honorably shelved, or buried 'deeper than e'er plummet sounded' out of the school-

* See "Conflict of Studies," by Todhunter, London, 1873.

† Nature, Vol. 4, 1871, p. 486.

boy's reach . . . ." The reform forces finally organized themselves, in 1871, into the "Association for the Improvement of Geometrical Teaching" (A. I. G. T.).

It is a curious circumstance that England's great mathematician, Arthur Cayley, opposed this reform movement. His admiration for Euclid was so ardent that he even expressed a preference for the original treatise without Simson's additions. In the opinion of Langley, Cayley "overshot the mark and his opposition told in favor of the Association."*

The second "Report" of the A. I. G. T. recommended practical exercises in geometrical construction, easy originals and numerical examples. Two years were given to the preparation of the Geometrical Syllabus on proportion. A double syllabus was prepared: A Syllabus on Geometric Constructions, and a Syllabus on Plane Geometry. Most of DeMorgan's suggestions† on the revision of the fifth book of Euclid were adopted.

This Society, after long labors, finally issued a substitute text, "The Elements of Plane Geometry." This was not used at home, but was used with success in the British colonies. Klein expresses himself, as follows, in regard to it: "This is essentially merely a smoothed down and polished presentation of the first six books of Euclid's elements; thus the rough places at the beginning of the first book . . . are removed by a consistent use of the concept of motion, but in general the sequence and the contents of Euclid are adhered to, in deference to the examinations. It is therefore only a tame reform, that is here attempted; nevertheless, it has met with sharp opposition by the adherents of the old English system. As proof of this, I refer to an amusingly written book of Dodgson, 'Euclid and His Modern Rivals.'" Here Euclid comes out victorious, and all reformers, particularly Legendre and members of the A. I. G. T., are put to the rout. J. M. Wilson's "Elementary Geometry," 1st edition, 1869, came in for a large share of the criticism. At Oxford, where Dodgson had given instruction in geometry for many years, this same Wilson had, at one time, read a critical paper before the Mathematical Society, on "Euclid as a Text-Book of Elementary Geometry." Wilson was a prime mover in the organization of the A. I. G. T.

* Fifth Report of the A. I. G. T., p. 21.

† "Companion to the British Almanac," 1849, pp. 5–20.

Since about 1870, many editions of Euclid have been printed containing revisions with the object of better adapting Euclid to school use. They exhibit all possible gradations of departure from the original text. There appeared sequels to Euclid like that of F. Casey. Professor Klein expresses himself in regard to these as follows: "The necessity has been felt to consider modern research, going beyond Euclid; this has been done by pressing it by force into the rigid Euclidean form, whereby a good part of the modern spirit is, of course, lost."*

During thirty years the A. I. G. T. appeared to have accomplished comparatively little. It had secured the concession that proofs different from Euclid's shall be accepted in examinations and had brought about a sentiment favoring some modification and enrichment of the Euclidean text. In reality, it had accomplished much more, for it had prepared the way for the great agitation of 1901, known as the Perry Movement, which called for a complete divorce from Euclid. The discussion of the teaching of mathematics at the Glasgow meeting of the British Association marks an epoch. The following are suggestions and criticisms that were contained in Perry's Syllabus.†

1. Experimental geometry and practical mensuration to precede demonstrative geometry. Use of squared paper. Rough guessing at lengths and weights to be encouraged.

2. Some deductive reasoning to accompany experimental geometry.

3. More emphasis on solid geometry; this subject has been postponed too long.

4. Adoption of co-ordinate representation in space.

5. The introduction of trigonometric functions in the study of geometry.

6. Emphasis upon the utilitarian parts of the subject.

7. Examinations conducted by any other examiner than the pupil's teacher are imperfect examinations.

In criticism of previous practices, Perry held that a boy should be educated through the experiences he already pos-

* Klein, p. 447.

† "Discussion on the Teaching of Mathematics," edited by John Perry, 1901, p. 97.

sesses, and should be allowed to assume the truth of many propositions. He held that the teacher must recognize that boys take unkindly to abstract reasoning. He criticized Oxford because, for the pass degree there, two books of Euclid must be memorized, even including the lettering of figures, no original exercises being required. In the discussion that followed, all favored the preliminary experimental course and some advocated a second experimental course to accompany Euclid. Hudson and Forsyth still believed in maintaining the Euclidean sequence of theorems. Minchin declared Euclid's order bad. S. P. Thompson and MacMahon favored the retention of Euclid. Miall did not see why we should have a recognized geometry any more than one arithmetic, or one trigonometry. Minchin, Magnus, Pressland, Workman, and Lamb declared themselves against Euclid as a text-book.

The immediate result of Perry's address of 1900, at Glasgow, was the appointment of two committees, one of the British Association and the other of the Mathematical Association. The former committee confined its work to the more general aspects of geometrical teaching. The latter, which was composed mainly of schoolmasters, formulated a set of detailed recommendations, which were published in the *Mathematical Gazette* of May, 1902. They include an experimental introductory course, requiring the use of instruments, practical measurement and numerical work. In the formal study of geometry is recommended the retention of Euclid as a framework, the admission of hypothetical constructions, definitions not to be taught *en bloc,* the omission of incommensurables in the ordinary school course, and the use of algebra in the treatment of areas.

The Perry laboratory method has led to the preparation of some severely practical works, but as Lodge says, Perry "over-emphasized fact divorced from principles." A middle ground has met with greater favor. The plans recommended by the Mathematical Association have been embodied very successfully by Godfrey and Siddons in a text-book entitled, "Elementary Geometry, Practical and Theoretical" (Cambridge University Press, 1904). The recommendations of the Mathematical Association have met with favor among teachers, and the general

effect has been beneficial. A circular issued in 1908–1909, by the Board of Education, on "The Teaching of Geometry and Graphic Algebra," showed the wide departure made since the beginning of the twentieth century. We quote two sentences: "Axioms and postulates should not be learnt or even mentioned." "It should be frankly recognized that unless the power of doing riders has been developed, the study of the subject is a failure."*

The greatest obstacle to reform in England has been the system of examinations. After thirty years of failures the Mathematical Association, at last, has been remarkably successful in persuading examining bodies to give up their insistence upon Euclid, and now Euclid's proofs and arrangement are no longer required by the universities. "Any proof of a proposition will be accepted which appears to the examiners to form a part of a logical order of treatment."

The United States of America.

During the seventeenth century, arithmetic and geometry received some attention in the last year of the college course at Harvard College. In 1726 Alsted's "Geometry" is mentioned as a text-book studied by Harvard seniors, but as soon as geometry came to receive serious attention in American colleges, Euclid became the text used. The first mention of Euclid that we have seen at Yale is in 1733; at Harvard, in 1737. In the latter part of the eighteenth century, geometry was taught to lower classmen. According to a member of the Harvard class of 1798, "the sophomore year gave us Euclid to measure our strength." In 1801 Professor Webber said, "A tutor teaches in Harvard College Playfair's 'Elements of Geometry.'"

In 1813 the "Analytical Society" was formed at Cambridge in England, which aimed to encourage in Britain the rigorous study of French higher mathematics. The influence of this movement reached the United States. In about ten years American teachers began to adopt French texts. Collateral events at West Point had the same tendency. There elementary mathematics was taught from 1808 to 1810 by F. R. Hassler, who was a graduate of the University of Berne in Switzerland.

* Nature, Vol. 80, 1909, May 27, p. 374.

In 1817 Crozet, of the Polytechnic School in Paris, introduced descriptive geometry into West Point.

In 1819, John Farrar, of Harvard, brought out a translation of Legendre's Geometry, which, with translations made by him of other French and Swiss texts on mathematics, were at once widely adopted in the leading American colleges. American teachers were willing to turn to the French, not only for works on the calculus and celestial mechanics, but also for books on elementary mathematics. So it came about that Euclid was replaced by Legendre. In 1828 Charles Davies, professor at West Point, brought out an edition of Brewster's translation of Legendre's Geometry. Davies did not enunciate propositions with reference to and by the aid of the particular diagram used for the demonstration, and to that extent returned to the method of Euclid. Davies' edition became widely popular under the name of "Davies-Legendre," and was much used in the United States as late as the '70's.

One of the earliest American geometries worthy of note was that of Benjamin Peirce. The Harvard catalogue of 1838 announces that Freshman take Peirce's Geometry. Peirce favored the use of infinitesimals and also the use of the term *direction*, a concept probably first used in this country by a Harvard teacher named Hayward in his geometry of 1829. Peirce's text did not become widely popular, for, like his other elementary books, it was too condensed for immature students. In 1843 or 1844, Harvard first made geometry a requirement for admission to College.

In 1851, Professor Elias Loomis, of Yale, issued a geometry which was revised in 1871. Loomis came under French influences as a student in Paris. In the second edition of his text he says: "The present volume follows substantially the order of Blanchet's Legendre, while the form of demonstrations is modeled after the more logical method of Euclid." It has been said of American writers, that while they have given up Euclid, they have modified Legendre's geometry so as to make it resemble Euclid as much as possible. This applies to Loomis with greater force perhaps than to any other author.

In 1871 Professor Olney, of the University of Michigan, published a "Geometry" under two main heads:

I. *Special or Elementary Geometry,* comprising (1) Empirical Geometry, (2) Demonstrative Geometry, (3) Original Exercises in the Application of Algebra to Geometry, (4) Trigonometry.

II. *General Geometry* (Plane Loci).

Olney was a self-educated man. He was a great teacher and had original ideas about teaching. It is said that he was prevented by his publishers from departing very far from the traditional classification. His ideas were novel and forecasted in many ways the present tendencies in mathematical teaching. His geometry shows that he attempted to correlate the various mathematical topics and to introduce applications to everyday affairs. Olney's books were used quite extensively in the Middle West, but acquired no firm foothold in the East.

Just before the death of William Chauvenet, in 1870, appeared his "Geometry," the only elementary book he wrote. Closely following French models, exhibiting a wonderful ease and grace of style, Chauvenet produced a remarkable book, which was used in many of the best schools. He included as a part of the work, an introduction to modern geometry. Perhaps no work on geometry ever published in the United States has been so highly respected as this.

In 1878 appeared the "Geometry" of G. A. Wentworth, which is still in use. We omit all discussion of it, as also of later books which have been published in this country.

The researches on non-Euclidean geometry, begun in the eighteenth century in Italy and Germany, and brought to fruition in the early part of the nineteenth century, did not produce appreciable effect upon the teaching of elementary geometry until the last quarter of the nineteenth century. It was in 1867 and 1868 that Baltzer, Battaglini, Grunert, and Hoüel brought Bolyai and Lobatchevsky to the attention of the mathematical public at large.

The new ideas have not affected the teaching of elementary geometry except in some of the definitions and postulates. They have assisted in the rejection of the definitions, "parallel lines are lines everywhere equally distant," and "parallel lines are straight lines which have the same direction." They have shown the futility of "proving" the parallel-postulate and have

led to the use of the word "axiom," not as a "self-evident truth," but as a synonym for "postulate."

In conclusion, we note that, with the beginning of the twentieth century, England began once more to influence the teaching of geometry in the United States, through the so-called "Perry movement," and that Germany, which at no time during the nineteenth century affected geometrical teaching in America, makes itself felt at the present time through the pupils of Klein and Hilbert and through the international movement towards reform in the teaching of mathematics, headed by Klein.

The Committee recommends the foregoing historical sketch to the careful consideration of teachers of geometry. Special attention is called to the age-long contest between the extreme formalists and the extreme utilitarians. The committee stands for neither extreme position. It recommends a reasonable attention to exercises in concrete setting, such, for instance, as simple problems involving the trigonometric ratios in connection with similar triangles, or such applications as those shown on pages 98–102 of this report. But in so doing it does not recommend diminishing attention to the logical side of the subject, but rather a quickening of the logical sense through a more rational distribution of *emphasis* which will make for economy of both time and mental energy in mastering the standard theorems and leave opportunity for a broader view of the subject in its concrete relations. See Sections D and E.

SECTION B. LOGICAL CONSIDERATIONS.

Axioms.

(a) **Nomenclature.**—The best historical usage distinguishes between Axioms (Euclid's "Common Notions") and Postulates (Euclid's "aitemata" or "requests") by including in the former certain general statements assumed for all mathematics, and in the latter certain specifically geometric concessions. These names and this distinction are now in general use and there seems to be no good reason for attempting to change them.

However, teachers who may wish to use the single term "assumptions" to cover both, or to use the term "axiom" to mean any proposition whose truth is postulated, thus making axiom and postulate synonymous, should be free to do so.

(b) General Nature.—It is evident that strict mathematical science would lead us to seek and to recommend an "irreducible minimum" of assumptions, while educational science leads us to see that such a list would be unintelligible to pupils and therefore unusable in the schools. Since we cannot recommend the adoption of a set of assumptions along the Hilbert line, we therefore lay down the general line of axioms and postulates needed in geometry, without insisting upon an exact list or upon any particular phraseology.

(c) General List of Axioms.—

As to the nature of the quantities, positive quantities are to be understood. When the negative quantity enters into elementary geometry it is in the discussion of propositions and not in cases in which the axioms are directly employed. For example, it is desirable not to confuse beginners in geometry by the question of dividing unequals by negative numbers.

Operations upon equal quantities.—It should be stated, preferably in a series of axioms, that if equals are operated upon by equals in the same way, the results are equal; *i. e.,* if $a=b$ and $x=y$, then $a+x=b+y$, $a-x=b-y$ (where $a>x$), $ax=by$, etc.

Operations upon unequal quantities.—It should be stated that if unequals are operated on by equals in the same way, the results are unequal in the same order; *i. e.,* if $a>b$ and $x=y$, then $a+x>b+y$, etc. These various cases enter into elementary geometry, and this assumption should be stated in such a manner that the student can easily refer to it in his work.

There is also the assumption that if unequals are added to unequals in the same order the sums are unequal in the same order, and that if unequals are subtracted from equals the remainders are unequal in the reverse order, these being the only ones relating to inequalities that are needed in elementary geometry.

As to substitutions.—In geometry it is continually necessary to make use of the assumption that a quantity may be sub-

stituted for its equal in an equation or in an inequality. Often this assumes the common form that "quantities that are equal to the same quantity are equal to each other." The committee recommends this axiom.

Inequality among three quantities.—It is necessary to say in geometry that if $a > b$ and $b > c$ then $a > c$, and an axiom to this effect is necessary.

The whole and its parts.—Although the definition of "whole" might be given in such a manner as to render unnecessary the usual axiom, it seems advisable to make the statement in the ordinary form.

It is to be understood that in applying these axioms to geometric magnitudes, the letters used refer to the ***numerical measures*** of such magnitudes, and as such belong to arithmetic and algebra, thus giving the theoretical basis for the correlation of geometry with these subjects.

Summary.—Axioms covering the above points are of advantage in the practical teaching of geometry, but this committee has no recommendation to make as to order or phraseology. They may be summarized as follows:

If $a = b$ and $x = y$, then

(1) $a + x = b + y$. (3) $ax = by$.

(2) $a - x = b - y$ $(a > x)$. (4) $a/x = b/y$.

(5) $a^n = b^n$ and $\sqrt[n]{a} = \sqrt[n]{b}$, where n is a positive integer.

(6) If $a > b$ and $x = y$, then $a + x > b + y$, $ax > by$, $a/x > b/y$, and if $a > x$, then $a - x > b - y$.

(7) If $a > b$ and $c > d$, then $a + c > b + d$, and if $x = y$, then $x - a < y - b$.

(8) If $a = x$, and $b = x$, then $a = b$.

(9) If $x = a$, we may substitute a for x in an equation or in an inequality.

(10) If $a > b$ and if $b > c$, then $a > c$.

(11) The whole is greater than any of its parts, and is equal to the sum of all its parts.

(d) General List of Postulates.—

(1) *One straight line and only one can be drawn through two given points.*

Corollary 1. *Two points determine a straight line.*

Corollary 2. *Two straight lines can intersect in only one point.*

(2) *A straight line-segment may be produced to any required length.*

This includes one postulate and one problem of Euclid, and so manifestly depends upon the simplest uses of straight edge and compasses as to be a proper geometric assumption.

(3) *A straight line is the shortest line between two points.*

(4) *A circle may be described with any given point as a center and any given line-segment as a radius.*

(5) *Any figure may be moved from one place to another, without altering its size or shape.*

(6) *All straight angles are equal.*

This and the following corollaries may be included among the theorems for informal proof under Section E.

Corollary 1. *All right angles are equal.*

Corollary 2. *From a point in a line only one perpendicular can be drawn to the line.*

Corollary 3. *Equal angles have equal complements, equal supplements, and equal conjugates.*

Corollary 4. *The greater of two angles has the less complement, the less supplement, and the less conjugate.*

The above axioms and postulates may be recommended for use as soon as the formal proof of propositions is begun, the postulate of parallels being introduced when needed, as follows:

Postulate of Parallels. *Through a given point one line and only one can be drawn parallel to a given line.*

The question of limits is considered later. It it not deemed desirable to postulate explicitly the existence of such concepts as point, line, and angle, nor to assume that a line drawn through a point in a triangle must cut the perimeter twice, nor to add a postulate of continuity. It is well, however, for teachers to mention that such assumptions are always *tacitly* made.

In any case the committee feels that a certain amount of care should be taken in fixing the location of points and lines and proving that lines intersect, when the accuracy of the proof in question might be affected by ignoring such details.

DEFINITIONS.

(a) New Terms.—

(1) *General principle.*—It is unwise for individual teachers or writers to introduce terms beyond those actually in common

use in geometry, or to change the accepted meaning of common terms, unless there seems to be a very definite advantage in the new term and an unquestionable sanction in the mathematical world. In particular, the substitution of a new term for an old one, to denote the same concept, is undesirable.

(2) *Type of terms that may safely be added* to those of the older elementary geometry: *congruent,* because this is so widely used both here and abroad, and because it avoids the loose use of *equal* and the long forms of *identically equal,* and *equal in all their parts.*

(3) *Types of terms that may safely be dropped: scholium,* because this has been so generally abandoned, and because it is unnecessary; *mixed line,* an antiquated term of no value in elementary geometry. Other such terms are *trapezium* and *rhomboid.*

(4) *Type of terms that seem of too doubtful advantage* to be recommended definitely by this committee, teachers being left free to use them if they desire, the terms thus being given an opportunity to make their way if they possess real merit: *ray,* a term that has abundant sanction in higher geometry, but may be dispensed with in elementary work. Other such terms are *mid-join* (for median), *cuboid* (for rectangular parallelopiped), *n-gon* (for polygon of *n* sides), and *sect* (for segment of a straight line).

(5) *Types of terms that are used with a different meaning in higher geometry,* and that may properly be used in elementary geometry with the more recent signification: *circle,* as meaning the line, which is, indeed, the primitive Greek meaning; *circumference,* as meaning the length of the circle—these usages requiring a redefining of *segment of circle, semicircle, area of circle,* and other obvious terms.

The committee recognizes also the tendency to unify the usage of such terms as *polygon* and *sphere* in elementary and higher geometry. This tendency should be encouraged. In any case, it is essential that the pupil should understand clearly what the terms mean in the statements and proofs of the propositions.

(b) Symbols.—

(1) *General principle.*—No symbols should be recommended beyond such as are already in wide use in elementary geometry,

and any that are unnecessary or are not generally accepted should be abandoned. The elaboration of personal symbolism, sometimes to the point of eccentricity, is such as to be cumbersome in the mathematics of the present.

(2) *Recommendations.*—The committee feels that the common symbols of algebra, most of which are known to the pupil beginning geometry, and such obvious symbols as those for perpendicular, triangle, circle, square, and parallel, are all that are needed in a course in elementary geometry, and that it is unnecessary to specify these symbols in detail. It appears that there is no generally received symbol for congruence, the symbols $=$, $\equiv$, and $=$ all being in use, and it seems best to recognize this fact, leaving teachers at present to decide the question for themselves. In due time a general consensus of opinion may lead to some definite usage, and it is the feeling of the committee that the second symbol given is a desirable one.

(c) Distribution of Definitions.—The committee recommends that new terms be taught when the time arrives for using them. This allows a teacher to use a book in which the definitions are massed or one in which they are scattered, but it encourages teaching them on the latter plan. It is recognized that the massed plan has the advantage of a dictionary arrangement, and this is a plan that a text-book writer might reasonably adopt, but it is not a plan to be followed in the actual teaching of the terms.

(d) The Defined and the Undefined.—The attention of teachers is called to the fact, now coming to be well recognized, that certain terms in geometry must be looked upon as *undefined.*

Certain concepts are so elementary that no simpler terms exist by which to define them, although they can easily be explained. For example, *point, line, surface, space, angle, straight line, curve.* The committee recommends that teachers give more attention to instilling a clear concept of such terms and none to exact definition.

On the other hand the committee recommends the careful definition of readily definable terms, where these definitions are parts of subsequent proofs, such definitions to be memorized exactly or in their essentials. For example, *right angle, square, isosceles triangle, parallelogram.*

There is a further class of easily defined terms, where the definition is not made the basis of a proof, and it seems obvious to the committee that the memorizing of the exact wording of such definitions is not a wise expenditure of time. Such terms are *hexagon, heptagon, re-entrant angle, concave polygon,* etc.

(e) The Form of Definition.—A definition may begin with the term defined, as in a dictionary; or it may close with the word defined; or it may at times contain the word in the midst of the sentence. The committee feels that it is of no moment which of these forms is taken, or that the definition be embodied in a single sentence. A definition that is to be memorized as the basis of a proof should be as nearly scientific as the powers of a beginner in geometry will justify, containing only terms that are simpler than the term defined, not being tautological, and being reversible—but further than this it seems unwise to attempt to specify the form of a definition.

Informal Proofs.

(a) Justification.—It is not pretended that elementary geometry is a perfect piece of logic. In general, the modern departures from Euclid have sacrificed logic for other ends, and even Euclid's "Elements" was not without numerous logical imperfections. That is to say, it has always been considered justifiable to sacrifice logic to a greater or less degree. The principle is that a logical sequence should be maintained, and formal proofs of propositions necessary to the sequence should be required, so far as this is consonant with the educational principle of adapting the matter to the mind of the learner. Now in many cases it happens that an informal and confessedly incomplete proof is more convincing to a beginner than a formal and complete one, and is less discouraging because it postpones the minor and seemingly unimportant steps to a time when their importance may be appreciated and the proofs understood.

(b) Types.—To be specific, the following are types of propositions that are better passed over by the beginner without formal statement, being introduced at the proper points in the development, or with informal proof, than proved in the euclidean fashion:

If one straight line meets another the sum of the two adjacent angles is a straight angle, and conversely (and related propositions);

All straight angles are equal (a proper postulate with related corollaries);

Two straight lines can intersect in only one point;

A straight line can have but one point of bisection (and the related case for angles);

The bisectors of vertical angles lie in one straight line;

Polygons similar to the same polygon are similar to each other;

If one angle is greater than another, its complement is less than the complement of the other (and related propositions);

A straight line can cut a circle (circumference) in two points only;

Circles of equal radii are equal (and related statements);

All radii of the same circle are equal (and similarly for diameters);

A circle can have but one center;

And propositions relating to the conditions under which two circles (circumferences) intersect.

It should be understood that these propositions are merely types, and that others of the same type may be treated in the same way, as specified in Section E of this report.

(c) Experience of Other Countries.—It is the experience of all countries where Euclid is not taught that good results follow from the use of a reasonable number of such informal proofs. The German and Austrian text-books are especially given to such procedure, and the results seem to have been favorable rather than otherwise. The number of propositions formally proved in a German text-book is notably less, for example, than in a corresponding French text-book.

(d) Dangers.—It is evident, however, that we may easily go to a dangerous and ridiculous extreme in this matter. With all of the experiments at improving Euclid the world has really accomplished very little except as to the phraseology of propositions and proofs; the standard propositions remain, and if geometry has any justification, apart from its kindergarten aspect (which requires but a short time), most of these proposi-

tions will continue to be proved, and should continue to be proved. These propositions, whether in the Euclid or Legendre arrangement, number in the neighborhood of 160 for plane geometry. Of this number upwards of one hundred must receive formal proof in any well-regulated course in geometry.

TREATMENT OF LIMITS AND INCOMMENSURABLES.

(a) Present Status.—It is generally agreed that the present treatment of this subject is open to two objections, (1) it is not sufficiently understood by the student to make it worth the while, and (2) it is not scientifically sound.

(b) Remedies Proposed.—Corresponding to the two defects mentioned two remedies have been proposed, (1) to make it less formal and technical, so that it shall be better understood, and (2) to abandon the incommensurable case altogether in secondary education.

(c) The Position of This Committee.—This committee recommends that in elementary geometry the nature of incommensurables and limits be explained, but that the subject no longer be required for entrance to college or be included in official examinations. It recommends that the schools treat the subject as fully beyond this point as circumstances seem to demand, and to this end reference is made to the syllabus given in Section E.

The prime object is to relieve the schools of the necessity of teaching the subject, while leaving them free to do so if they wish.

TIME AND PLACE IN THE CURRICULUM.

(a) Conventionally.—At present, in America, plane geometry is generally taught in the tenth school year (not counting the kindergarten).

In the East it is completed in the eleventh school year, and in the West solid geometry is completed in the eleventh or twelfth year. In spite of all the discussion about constructive geometry (intuitive, metrical, etc.) in the first eight grades, carried on in the past half century, no generally accepted plan has been developed to replace the old custom of teaching the most necessary facts of mensuration in connection with arithmetic. We have, therefore, at this time, algebra in the ninth

school year, plane geometry in the tenth, and algebra and geometry in the eleventh and sometimes in the twelfth.

(b) Changes Suggested.—Certain changes in this conventional plan have been suggested.

(1) To provide for preliminary (inductive, constructive, observational) work in geometry in the elementary grades. This topic is discussed in Section C of this report.

(2) To precede the work in plane geometry by some definite work in geometric drawing. Attention may be called to the fact that the recent great advance in art education has had one disadvantage from the standpoint of geometry, in that geometric drawing has been abandoned, and that therefore some little work in handling compasses and ruler must now form part of the first steps in this subject.

(3) To unite geometry and algebra, or geometry and trigonometry. This committee does not feel that the experiments along this line, which have been made in only a few schools, have been sufficient to determine whether or not geometry should run parallel with algebra in the ninth, tenth and eleventh school years.

(c) Position of This Committee.—This committee recommends that plane geometry be assigned not less than one year nor more than one and one half years in the curriculum, being preceded by at least one year of algebra except where the individual teacher desires to carry it along with algebra.

It should be distinctly understood that owing to the condition of unrest in the entire field of secondary education it is at present impossible to give any final advice along any of these lines of change. It is probable that many of the readjustments now under general discussion will influence every high school curriculum in the course of time. It is also possible that some of the proposed changes will be adapted by the different types of secondary schools to their own needs, and that they will receive greatly varying emphasis in different localities. A certain amount of experimentation will undoubtedly be necessary to test the feasibility of some of the proposed plans. Great care should be taken to make all such experimentation with due regard for all that was good in the past, so that the new curricula may be the result of evolution, and not of revolution.

The most noteworthy tendency in secondary education is the desire for more organic teaching and hence the desire for more time. This tendency finds its most significant expression in the movement toward a *six-year curriculum.* It is undoubtedly true that in a six-year curriculum many of the problems of correlation would be brought nearer to a solution, that many difficulties arising from the present tandem system would disappear, and that mathematics would be given a place in the curriculum more nearly commensurate with its importance.

For a brief account of the six-year curriculum the reader is referred to the book of Hanus entitled "A Modern School," published by the Macmillan Company; and also to the Proceedings of the National Education Association for 1908.

Purpose in the Study of Geometry.

(a) Historical Review.—Geometry was originally, as its name indicates, purely a practical subject. This phase of its history remains in the work in mensuration in arithmetic to-day. It then became a philosophical subject, connecting with mysticism in the Pythagorean school, being put upon a more solid scientific basis by the Platonists, and being crystallized by Euclid about 300 B. C. Since that time the formal side has dominated. But this formal side has been attacked time after time, by the astrologers and mystics, by the cathedral builders of the Middle Ages, strongly by the French writers of the seventeenth and eighteenth centuries, recently by an extreme school in England, and at present in a less formidable fashion in our own country. The results of these attacks in so far as they have meant the abandoning of formal proofs have been futile.

(b) The Practical Side.—In the high school geometry has long been taught because of its mind-training value only. This exclusive attention to the disciplinary side may be fascinating to mature minds, but in the case of young pupils it may lead to a dull formalism which is unfortunate. On the other hand those who are advocating only a nominal amount of formal proof, devoting their time chiefly to industrial applications, are even more at fault. The committee feels that a judicious fusion of theoretical and applied work, a fusion dictated by common sense and free from radicalism in either direction, is necessary.

As to the nature of the applications, the committee feels that there are several types of genuine problems, but that many of the so-called real applications either are too technical to be within the grasp of the young beginner, or represent methods of procedure that would not be followed in real life. Moreover, it should be remembered that the very limited time devoted to plane geometry (usally a single year) renders it impracticable to introduce many of the applications that might be desirable if the time were not so restricted.

(c) The Formal Side.—No reference to the applications of geometry is to be construed to mean that the committee feels that the formal side should suffer, or that geometry is wanting in a distinct disciplinary value. A formal treatment of geometry, to about the traditional extent, is necessary purely as a prerequisite to the study of more advanced mathematics, and still more because such treatment has a genuine culture value, for example, in assisting to form correct habits in the use of English.

Certain writers on education have claimed that geometry has no distinctive disciplinary value, or that the formal side is so intangible that algebra and geometry should be fused into a single subject (not merely taught parallel to each other), which subject should occupy a single year and be purely utilitarian. These writers fail to recognize the fundamental significance of mathematics in either its intellectual or its material bearing.

(d) Claims for Geometry.—Among the claims in behalf of geometry the committee would emphasize the following:

Geometry is taught because of the *pleasure* it gives when properly presented to the average mind.

Geometry is taught because of the *profit* it gives when properly presented. For example:

(1) It is an exercise in logic, and in types of logic not generally met in other subjects of the school course, and yet types which occur in geometry in unusually simple setting and which are easily carried over into the actual affairs of life. Closely connected with the logical element is the training in accurate and precise thought and expression and the mental experience and contact with exact truth.

This logic may be no more practical than literature or art

or any other great branch of learning, but its general effect on the human mind has been doubted by such a small number of scholars as to render it worthy of the highest confidence.

(2) The study of geometry leads also to an appreciation of the dependence of one geometric magnitude upon another, in other words to the tangible concept of *functionality*.

(3) The study of geometry cultivates space intuition and an appreciation of and control over forms existing in the material world, which can be secured from no other topic in the high school curriculum.

(4) The value of the applications of geometry to mensuration and the satisfaction derived by the pupil in verifying the formulas of mensuration already met by him in arithmetic are well recognized by all teachers.

If we had to justify the position of any other subject in the curriculum, history, rhetoric, geography, biology, etc., it is doubtful whether equally specific and cogent reasons could be found. If we were to dismiss geometry with a few practical lessons, much more should we be compelled to dismiss most other subjects in the curriculum with the same treatment.

Historical Notes.

Of the stimulating effect of occasional bits of historical information given by the teacher or the text-book there can be no question. There is plenty of material to be found in the well-known elementary histories of the subject. The discoverers of particular propositions are known in a few cases, and the general story of the subject, told informally as the pupil proceeds in his study, adds a human interest that is valuable. Portraits of famous mathematicians may be recommended for the schoolroom.

Points Relating to Solid Geometry.

(a) **Axioms and Postulates.**—The list of axioms already given need not be increased, but the following postulates may be added:

(1) *One plane and only one can be passed through two intersecting straight lines.*

Corollary. *A plane is determined by three points not in the*

same straight line, by a straight line and a point not in it, or by two parallel lines.

This postulate, which is the analogue of the first postulate in plane geometry, may also be given as a theorem for informal proof.

(2) *Two intersecting planes have at least two points in common.*

(3) *A sphere may be described with any given point as center and any given line-segment as radius.*

It is tacitly assumed that the figures described in the course in solid geometry exist and can be made the subject of investigation; *e. g.*, the prism, pyramid, cylinder, cone, etc.

It may also be assumed, tacitly or explicitly, that the various closed solids have definite areas and volumes; *e. g.*, that a sphere has a definite volume which is less than that of any circumscribed convex polyhedron and greater than that of any inscribed convex polyhedron.

(b) Definitions.—Latitude is left to the teacher in regard to the use of such terms as prismatic space, cylindrical space, nappes of a cone, and some of the names suggested for a rectangular parallelopiped, which are convenient but not necessary in an elementary course.

After the analogy of the circle defined as a line, it is proper that the sphere be defined as a surface but the more common definition may be retained if desired.

(c) Purpose.—In solid geometry the utilitarian features play an increasingly important part. The mensuration involved in plane geometry is so simple as to be fairly well understood as presented in arithmetic. Solid geometry, however, offers a rather extended field for practical mensuration in connection with algebraic formulas. The subject is therefore particularly valuable for high school classes. A further application is found in the power afforded to visualize solid forms from flat drawings, a power that is essential to the artisan and valuable to everyone. The committee therefore summarizes the purposes of solid geometry as follows:

(1) To emphasize and continue the values of plane geometry, mentioned above;

(2) To present a reasonable range of applications to the field of mensuration;

(3) To cultivate the power of visualizing solid torms from flat drawings, without entering the technical domain of descriptive geometry.

SECTION C. SPECIAL COURSES.

(a) Courses for Different Classes of Students.—

One of the topics which this committee undertook to consider was that of different courses for various classes of students in the high schools.

After investigation, it is the belief of the committee that there should be no attempt to outline such courses. The syllabus as recommended in Section E may be altered in special cases by the omission of the theorems printed in small type and by increased emphasis upon theorems which admit of direct practical applications.

The preceding recommendation, together with the possible omission of solid geometry, would reduce the course to less than half the traditional length. It seems probable that no greater reduction would be desirable even for students in purely commercial courses, or indeed in any course in which formal geometry is a required subject.

(b) Preliminary Courses for Graded Schools.—

A portion of the report of this committee was to deal with preliminary courses to be undertaken in graded schools.

Recommendations.—It is of the utmost importance that some work in geometry be done in the graded schools. For this there are at least two very strong reasons. In the first place, geometric forms certainly enter into the life of every child in the grades. The subject matter of geometry is therefore particularly suitable for instruction in such schools.

Moreover, the motive for such teaching is direct. The ability to control geometric forms is unquestionably a real need in the life of every individual even as early as the graded school. For those who cannot proceed further this need is pressing; the direct motive involved compares very favorably with any other direct motive for work in the grades. For those who are going on to the high school, the development of the appreciation of geometric forms is almost an absolute prerequisite for any future work in geometry.

Informal Work.—It is quite obvious that no work of formal, logical character should be undertaken in the graded schools. The earliest work in geometry will doubtless be so informal that it will not constitute a separate course. Instruction in drawing, in pattern making, and in elementary manual training furnishes a basis for considerable geometric work even in the first grades of the primary school.

Such work as this should be encouraged, though no special outline of it can be given on account of its dependence upon other courses. The constructions for erecting perpendiculars, bisectors of angles, etc., can and should be given in connection with such manual training work as making boxes, patterns, etc., though no technical nomenclature need be used. In such work paper folding and the use of simple instruments should be encouraged, including the compasses, the ruler, and in later years the protractor and squared paper.

Mensuration.—In connection with arithmetic much geometric work may be taken up which is consistent with the child's real interests and life. Measurement may be introduced very early and the mensuration of simple forms such as the square, rectangle, and triangle need not be long delayed. After this, other geometrical forms and solids may be introduced under the head of mensuration even earlier than is now customary. In properly conducted schools, the students will become familiar at the same time with such figures as the circle, cube, sphere, etc., in manual training and in other elementary courses, such as nature study, geography, etc.

In the later grades practically all of the simple geometric forms will find their place in arithmetic under the head of mensuration, in drawing, and in manual training.

Work in the Higher Grades.—A special course in geometry in the graded school is desirable, if at all, only in the last grade or the last two grades. In such a course no work of demonstrative character should be undertaken, though work may be done to *convince* the student of the truth of certain facts; for example, by paper folding or cutting a variety of propositions may be made evident, such as the sum of the angles of a triangle is 180°, etc.

The theorem just named is typical of the theorems which the

student should know as *facts* before he leaves the graded school. Many others of the theorems printed in black faced type in the syllabus submitted herewith may be taught in this course without formal proof.

Theorems as Facts.—Emphasis should be laid upon the facts with which the student is already familiar through the work described above. The course should be regarded partially as a classification and a systematization of the knowledge previously acquired. Thus, simple geometrical forms should be brought up in the connection in which they have arisen in the student's past experience. In taking up constructions, explicit mention should be made of the previous work in which a given construction occurred, and further practical instances of the use of such constructions should be given.

Drawing to Scale.—Emphasis should also be laid on other work of a concrete nature which involves direct use of geometric facts. Thus the propositions concerning the similarity of triangles should be introduced by means of the drawing of figures to scale. Attention should be called to the case in which figures have been drawn to scale in the past. The usefulness and the necessity of the operation should be emphasized, and such applications as the drawing of house plans, the copying of patterns on a smaller scale, etc., should be given. The use of cross section paper for this purpose may be encouraged. Finally after the notions involved are very clear indeed, and after actual measurements have been made and reduced to scale the precise facts regarding similar triangles may be given. This should follow and not precede the work described above. The applications to elementary surveying should, if possible, be made in actual field work.

Models and Patterns.—The situation just described for similar triangles should be carried out so far as possible in other instances. Thus the important theorems on the measurement of angles can be illustrated in many ways. The Pythagorean theorem, without formal proof, can be illustrated and made real to every student by reference to the pattern forms in which it occurs, the calculation of distances, and other real applications. Finally the mensuration formulas can all be given. In the latter, concrete illustrations should abound and verification by

means of models and measurements upon them should be encouraged.

Forms of Solid Geometry.—Contrary to the traditional procedure, the forms of solid geometry should be emphasized even more than those of plane geometry, for they are more real and more capable of concrete illustration.

Not only should formal demonstration be avoided but also long lists of definitions which tend to confuse rather than enlighten. Definitions should be stated formally only *after* the concept is clearly formed in the student's mind. No axiom should be stated as such at any point, though frequent assumptions should be made without an attempt at proof. In all such cases care should be taken that the statements made seem reasonable to the student and no forward step should be taken until he is absolutely convinced of the truth of the statement.

Justification.—That such work is of vital value to the student can scarcely be doubted; that it is absolutely legitimate will probably be admitted by all interested in primary education. Its value, its real direct motives, its contact with life, the legitimacy of its subject matter exceed incomparably those of the traditional course in advanced arithmetic. At least, the course in arithmetic may be vitalized by a liberal infusion of such geometric work.

If such a course is not given in the grades—perhaps even though it is—a course of similar character but very much shorter may be given in the high school before formal work in demonstrative geometry is attempted. In any event it is desirable that the course in formal geometry should not proceed in its traditional groove until the teacher is assured that the ideas mentioned above are thoroughly familiar to the student.

SECTION D. EXERCISES AND PROBLEMS.

Distribution, Grading, and Nature of Exercises.

(a) Increasing Number of Exercises.—There has been a growing tendency in the last two decades to increase ***abnormally*** the number of exercises to be considered by each pupil under the following heads: (1) long lists of additional theorems (beyond the full set usually given in the texts), (2) long lists of problems of construction having at best remote connection

with any uses of geometry within reach of the ordinary high school pupil, (3) long lists of numerical exercises given in the abstract, that is, unrelated to any concrete situation familiar to the pupil or arousing his interest.

To give a single illustration of each:

(1) The squares of two chords drawn from the same point in a circle have the same ratio as the projections of the chords on the diameter drawn from the same point.

(2) To construct a triangle having given the perimeter, one angle and the altitude from the vertex of the given angle.

(3) Through a point P in the side AB of a triangle ABC, a line is drawn parallel to BC so as to divide the triangle into two equivalent parts. Find the value of AP in terms of AB.

(b) The Distribution of Exercises.—It is recommended that there should be treated in connection with each theorem such immediate concrete questions and applications as are available, and especially early in the course should such theorems be given as easily lend themselves to this class of exercises.

For example, in a treatment in which the theorems on congruence of triangles are placed early, there is the opportunity to bring in at once the simplest schemes for indirect measurement of heights and distances. Then later as similarity of triangles is taken up, there is the chance to recur to the same problems and let the pupil see how the principle adds power and facility in making indirect measurements. There is thus a progressive development in the facility for solving concrete problems along with the theory.

This principle can be carried out in many different lines. For example, in connection with triangles, circles, and squares, there are many applications immediately available and easily found in tile patterns, window tracery, grill work, steel ceiling patterns, etc. These afford fine exercises in construction early in the course, and are equally available later in the computation and comparison of areas. When such exercises are given they should be distributed as far as possible in connection with the theorems used in the construction and comparison of the figures involved.

However, only the simplest uses of the theorems can be shown in the immediate connection, both because of the space occupied

by them and the danger of interrupting the continuity of the theorems by too many exercises thrown in between them, and also because most of these applications make use of various different theorems, and hence must come after certain groups of theorems, thus making necessary occasional lists of problems and applications scattered through the various books, as well as sets of review exercises at the end of each book.

The whole question of distribution is thus to be determined by the relation of the problems and applications to the single theorems or groups of theorems to which they belong. The important question of emphasis in Section E of this report is best brought out by the grouping of many exercises around the basal theorems.

On the basis of distribution we have all extremes in the various texts, including: (1) The purely logical presentation, that is, the continuous chain of theorems with practically no applications in concrete setting in connection with them and almost none at the end of the books; (2) the same as the foregoing, except that the long sets of exercises are placed at the end of each book, where they loom up before the pupil as great tasks to be ground through, if, indeed, they are not omitted altogether; (3) the psychological presentation in which the more difficult exercises are either postponed to a later part of the course or are omitted altogether, and the easier ones are brought into more immediate connection with the theorems to which they are related.

The time and space made available by the third method of presentation provides an opportunity for the pupil to gain some acquaintance with the uses of the theorems as he proceeds and to become genuinely interested in the development of the subject. The committee strongly recommends this latter method of presentation. In expressing its disapproval of method (2), it is not to be understood that the committee objects to any text-book because it offers a large number of exercises, placed at the end of each book, from which the teacher is to make a selection. The objection to (2) should be clear from reading (3), which the committee approves.

(c) **The Grading of Exercises.**—Too much cannot be said in favor of a large number of simple cases rather than too many

difficult questions, especially early in the course, but also even throughout the secondary course in geometry.

The average high school pupil is not likely to become adept at proving difficult and abstruse theorems independently or in solving complicated problems. On the other hand, the rank and file are bound to become discouraged and hopelessly lost in the so-called "originals," unless the grading is carefully done, and steps of difficulty are kept down to a very reasonable lower limit.

The ideal treatment would seem to be: (1) To make a proposition appeal to the pupil as reasonable by simple illustrations, after which should follow the deductive proof; (2) to apply the theorem to more difficult situations, involving problems which the pupil regards as interesting and worth while. It is recognized that this ideal cannot be attained with reference to all the theorems of geometry but it is believed that it can be attained in very many cases; and, wherever this is possible, great interest and incentive are given to the pupil.

As a matter of fact, familiarity with the elementary truths pertaining to angles, parallelograms, and circles, when consistently tried out and seasoned by applications to numerous comparatively simple and interesting geometric forms suggested by figures which abound in concrete setting on every hand within reach of all, is usually of more value to the average pupil (and even to the better pupils) than is the study of a larger number of abstract theorems or problems through which they are often forced. Nevertheless, for the benefit of the brighter pupils, it is desirable that a few comparatively difficult problems be given, especially at the ends of the various books, or in a supplementary list.

(d) The Nature of Exercises.—This topic has been referred to under (a), (b), and (c). It is recognized that a fair proportion of the traditional exercises given in abstract setting should find a place in a course in geometry, but the committee believes that, in accordance with the common practice of the past twenty years, this class of exercises has been magnified and extended, especially with reference to the more difficult exercises, beyond the interest and appreciation of the average pupil.

The committee therefore recommends that a judicious selec-

tion of a reasonable number of abstract originals be made in order to leave time for an equally reasonable number of problems, particularly those with local coloring, stated in concrete setting.

Since ample lists of abstract originals are within easy reach of all teachers of geometry, it seems unnecessary to supply illustrations of such exercises. But as the teacher must generally depend upon his own initiative to supply problems in concrete setting, it seems desirable to indicate a few sources from which such problems may be obtained. The committee believes that a reasonable number of problems of this character creates an interest in the minds of the pupils that reacts strongly in augmenting his understanding and appreciation of the logical side of the subject. But it is not to be understood that the committee regards these problems as practical in the narrow sense of the word.

Sources of Problems.

(a) Architecture, Decoration, and Design.—Industrial design and architectural ornament are replete with details that may be used as a source of supply for geometry problems. These problems are of three kinds: (1) The problems involved in the construction of the figures themselves; (2) the demonstrations necessary to establish numerous relations which are visible to the mathematician and which must occasionally be assumed by designers; (3) problems in computation.

Among the industrial products that involve geometric ornament are tile and mosaic floors, parquetry, linoleum, oilcloth, steel ceilings, ornamental iron, leaded glass, cut glass, and the like. Figures for problems from these sources may be made from the cuts in trade catalogues.

Problems based on architectural ornament are largely from details of Gothic tracery and can be obtained only by a study of the buildings themselves or of the photographs of them that may be seen in architectural libraries. Gothic tracery is found in windows, in ornamental iron, in carved stone and wood on the outside and inside of buildings, on furniture, choir screens, rafters, and the like, that abound in medieval cathedrals and churches and in their modern imitations.

These problems have distinct advantages. In many cases their comprehension and solution require no technical knowledge beyond the elementary mathematics needed. These designs abound, are largely within the reach of pupils, and their use in the class room brings before pupils as nothing else can the beauty and widespread application of geometric forms. In them may be found applications of many topics of elementary mathematics and from them may be obtained numerous exercises of all grades, from the simplest to the most complex. By their use it is possible, therefore, to introduce anywhere in the work problems that are within the reach of the average pupil and appeal to him with a minimum of experiment, explanation, discussion, or previous special preparation.

(b) Problems of Indirect Measurement.—It should not be considered that the types of applications under (a) are relatively of greater importance than numerous others. Any application that adds interest to the study of rigorous geometry is of value. Of special interest are all simple means of effecting indirect measurements of distances, such, for instance, as the numerous applications of the congruence theorems and the theorems on similarity of triangles. Here the teacher will find much assistance in Principal Stark's "Measuring Instruments of Long Ago." Again in considering the isosceles triangle, the universal leveling instrument (aside from the spirit level) offers a number of applications. The form is that of an isosceles triangle bisected by a line from the vertex.

Many simple and interesting problems in indirect measurement are made available by the introduction of the trigonometric ratios, *sine, cosine,* and *tangent.* This can be done as soon as the theorems on similar triangles are known, and the computation by measurement of a two place table of natural functions at intervals of 5° affords, of itself, a fine drill in the application of these theorems, at the same time providing material for solving concrete problems of great interest to young pupils.

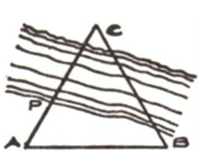
It may not be possible to find time for this in connection with the usual course in plane geometry. Schools that devote most of their time and effort to preparing pupils to pass entrance examinations for college would certainly find it difficult to

meet any added requirement. In view, however, of the omissions suggested by this committee and the readjustment of emphasis on basal theorems, time may be found, as the experience of an increasing number of teachers has shown. Where this can be done it constitutes an important step in the closer correlation of the subjects in elementary mathematics. In any case only the natural functions should be used, and the applications should be limited to those involving right triangles.

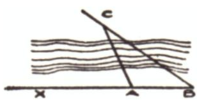
(c) **Other Sources.**—Problems may also be obtained from physics, mechanics, and other sciences, from engineers' and builders' manuals and works on carpentry and masonry, as, for instance, problems derived from various common forms of trusses and the construction of arches. But there is a danger in connection with problems from these sources, that aside from the geometry involved, they may contain technical terms and mechanical features unknown to the average pupil and not easily understood without more explanation and consequent distraction from the geometry itself than is warranted in the ordinary course. Problems of this class should be carefully tried out and sifted before being adopted for use.

(d) **Illustrative Problems.**—There are given below a few typical problems which are suggested for the purpose of making clear what the committee has in mind. Through simple problems of these types and many others which might be suggested much interest can be imparted to the study of demonstrative geometry, even though the problems be not practical in the strict sense of the word. The danger of using too many problems in any narrow field is, however, apparent.

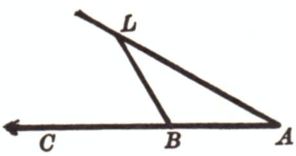
(1) The theorem regarding the angle sum in a triangle has a large number of applications. For example, to measure PC, stand at some convenient point A and sight along APC and (by the help of an equilateral triangle cut from pasteboard) along AB. Then walk along AB until a point B is reached from which BC makes with BA an angle of the equilateral triangle (60°). Then $AC = AB$, and since AP can be measured we can find PC. This is an example of a problem that adds interest to the work without being itself a practical application that would be used by a surveyor.



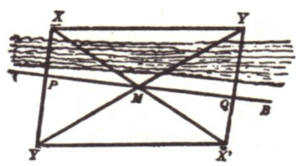
(2) A problem of the same nature is the following: To measure AC, first measure the angle CAX, either in degrees with a protractor or by sighting across a piece of paper and marking it down. Then walk along XA produced until a point B is reached, from which BC makes with BA an angle equal to half of angle CAX. Then it is easily shown that $AB = AC$.



(3) The sailor makes use of this principle when he "doubles the angle on the bow" to find his distance from a lighthouse or promontory. If he is sailing on the course ABC and he notes a lighthouse L when he is at A, and takes the angle A, and if he notices when the angle that the lighthouse makes with his course is just twice the angle noted at A, then $BL = AB$. He has AB from his log, so he knows the distance BL.

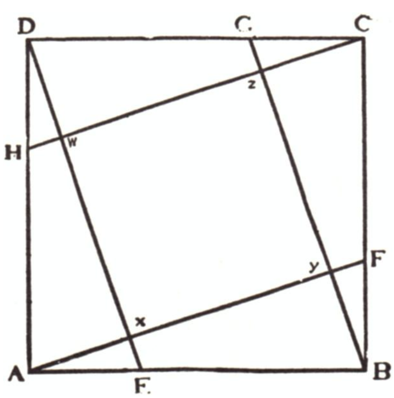


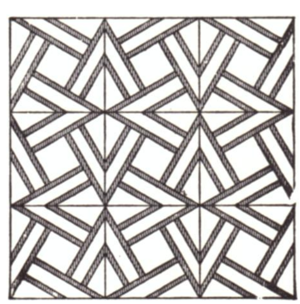
(4) To measure the line XY, when the observer is at A, we may measure any line AB along the stream. Then the observer may take a carpenter's square, or even a large book, and walk along AB until a point P is reached from which X and B can be seen along two sides of the square. Similarly the point Q may be fixed. Then by walking along YM to a point Y' that is exactly in line with M and Y and also with P and X, the point Y' is fixed. Similarly X' is fixed. Then $X'Y' = XY$.



(5) A field containing 9 acres is represented by a triangular plan whose sides are 12 in., 17 in., and 25 in. On what scale is the plan drawn? Conant.

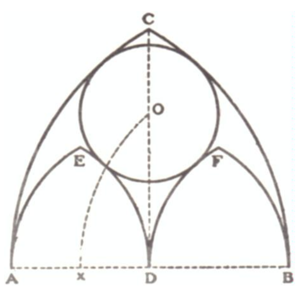
(6) Assuming the earth to be a sphere of which the radius is 3,960 miles, find the length of one degree of longitude at 60° north latitude, and compare its length with that of one degree of longitude at the equator.

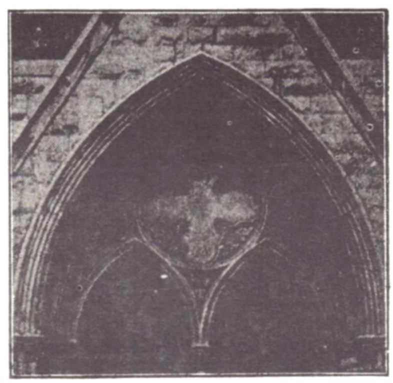




(7) $ABCD$ is a square. Equal distances AE, BF, CG and DH are measured off on the sides AB, BC, CD and DA respectively. If the lines AF, BG, CH and DE are drawn intersecting at Y, Z, W and X, prove that $XYZW$ is a square. If $AB = a$ and AE is $\frac{1}{3}$ of AB, prove that $AF = a/3\sqrt{10}$, $AX = a/10\sqrt{10}$, $XY = a/5\sqrt{10}$, $FY = a/30\sqrt{10}$; and that the area of $XYZW$ is $2a^2/5$.

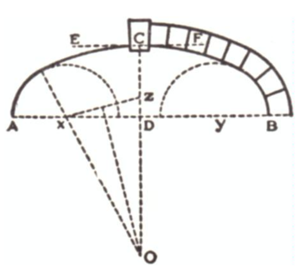
This figure is the basis of an Arabic design used for parquet floors. The solution involves both algebraic and geometric work in concrete setting.





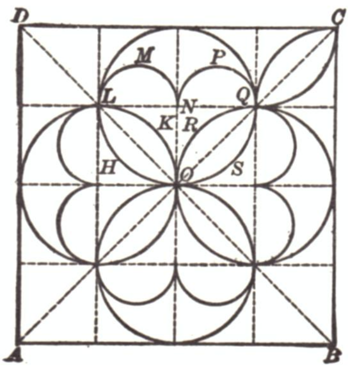
(8) ABC is an equilateral arch, and CD its altitude. A is the center of the arc BC and B the center of the arc AC. The equilateral arches AED and DFB are erected on AD and BD respectively. D is the center of arc AE and FB and A and B are centers of arcs ED and DF, each drawn with $\frac{1}{2}AB$ as radius. What is the locus of centers of circles tangent to CA and CB? To ED and DF? To AC and DF? To CB and ED? Construct a circle tangent to the arcs AC, CB, ED and FD.

This figure is the basis of a common Gothic window design. The solution involves the intersection of loci.

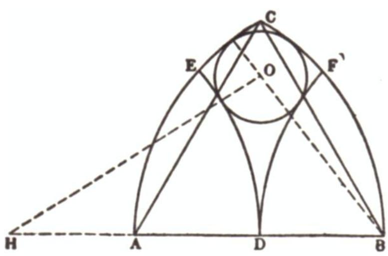


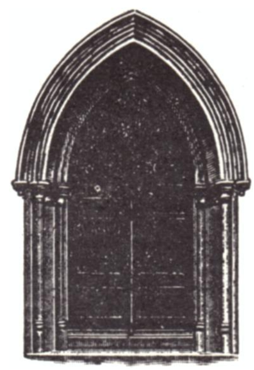
(9) CD is the perpendicular bisector of AB. Equal distances AX and BY are measured off on AD and BD respectively. EF is perpendicular to CD at C. Circles are drawn with X and Y as centers and AX and BY as radii. Construct a circle tangent to EF at C and to circle X. Prove that this circle is also tangent to the circle Y.

If CD is less than $\frac{1}{2}AB$ the part of this figure between lines CD and AB is one form of a three centered arch.



(10) In the drawing above, which is the basis of the mosaic floor design to the right, the circle with center N is inscribed in one of the squares whose side is SH. The arcs ORQ and OKL are drawn with the vertices of the square as centers and half the side as radius. The semicircles LMN and NPQ are drawn on LN and NQ as diameters. Find the areas of the various figures bounded by circular arcs within this square. Note the symmetry of the whole figure within the square $ABCD$.





(11) ABC is an equilateral triangle. A and B are centers of arcs BC and AC respectively. CD is the altitude of triangle ABC. Arcs DF and DE, constructed with radii equal to AB, are tangent to CD at D and intersect AC and CB respectively at E and F. Construct a circle tangent to the arcs DE, DF, AC and BC.

Suppose the problem solved. Let O be the center of the circle. Connect O with B, the center of arc AC, and with H, the center of arc DE. From triangles ODB and OHD the following equation is derived:

$$(s-r)^2-(s/2)^2=(s+r)^2-s^2,$$

where s is the length of AB and r is radius of the required circle.

This figure is the basis of a church window design. Many problems of this type may be easily obtained.

(12) A quarter mile running track has two parallel sides and semicircular ends. Each straight away section is equal in length to one of the ends. If the track measures exactly one-fourth of a mile at the curb, or inner edge, how much distance does a runner lose in running two feet from the curb? Six feet? What is the area of the track if it is 15 feet wide? What is the area of the enclosed field? What are the dimensions of a rectangular field sufficiently large to contain such a track? What will it cost at $2.00 per cubic yard to cover such a track with cinders to a depth of 2 inches? Pettee.

(e) References to Sources of Problems.—In connection with the recent search for real applied problems in elementary mathematics numerous bibliographies have been compiled to which reference is here made, as well as to a few other books, aside from current texts, which may be helpful to teachers. Concrete problems should be selected carefully and used wisely. Those which may appear to one class of pupils as real applied problems may seem highly abstract to another class. Probably few problems in the following lists would appear real to all pupils and yet all are likely to find increased interest in any problem which has a concrete origin.

Printed bibliographies.—

1. A list of 38 titles of books and 21 titles of trade journals, *School Science and Mathematics*, Vol. IX., No. 8, 1909, pages 788–798.
2. A more extended list of books on the whole range of applied problems, *School Science and Mathematics*, Vol. VIII., No. 8, November, 1908, pages 641–644.

   From this list Saxelby, Godfrey and Siddons, and Perry may be mentioned especially.
3. A comprehensive list of books and journals relating to the uses of geometry in architecture, decoration, and design in a forthcoming volume entitled "A Source Book of Problems for Geometry," by Mabel Sykes. Allyn and Bacon, Boston, 1912.
4. A vast bibliography of suggestive titles, with a classification and discussion of some phases of industrial problems by a committee of the National Education Association on "The Place of Industries in Public Education." *Proceedings* of the Association, 1910, pages 652–788.

Collections of problems.—

1. Real Problems in Geometry, *Teachers College Record*, March, 1909. A classification and discussion of types of applied problems, by James F. Millis.

2. Real Applied Problems in Algebra and Geometry, *School Science and Mathematics*. A collection begun in 1909 by a committee of the Central Association of Science and Mathematics Teachers. The work is still in progress. The problems collected up to November, 1909, have been classified and published in pamphlet form.

Other selected titles.—

1. "Lessons in Experimental Geometry," Hall and Stevens. The Macmillan Company, New York, 1905.
2. "Numerical Problems in Geometry," J. G. Estill. Longmans, Green and Company, New York, 1908.
3. "Mensuration," G. B. Halsted. Ginn and Company, Boston, 1910.
4. "Elementary Mensuration," F. H. Stevens. The Macmillan Company, New York, 1908.
5. "A Notebook of Experimental Mathematics," Godfrey and Bell. London, Edward Arnold, 1905.
6. "Elements of Mechanics," M. Merriman. John Wiley and Sons, New York, 1905.
7. "Shop Problems in Mathematics," Breckenridge, Mersereau and Moore. Ginn and Company, New York, 1910.
8. "Pocket Companion containing Tables," etc. Carnegie Steel Company, Pittsburgh, Pa., 1903.
9. "Leitfaden der Geometrie," Jahne and Barbisch. Vienna, 1907.
10. "Raumlehre für Mittelschulen," Martin and Schmidt. Berlin, 1898.
11. "Geometrie für die Zwecke des practischen Lebens," G. Ehrig. Leipzig, 1906.
12. "Mathematische Aufgaben," Schulze and Pahl. Leipzig, 1908.
13. "Cours abrégeé de Géométrie," Bourlet and Baudoin. Paris, 1907.
14. "Cahiers d' exécution de dessins géometriques," M. P. Baudoin. Paris.
15. "Geometria Intuitiva," P. Pasquali. Milan.
16. "Regole di Geometria Pratica," F. Aandreotti. Florence, 1897.
17. "The Power of Form Applied to Geometrical Tracery," R. W. Billings. London, 1851.
18. "Gothic Architecture in England," Francis Bond. London, B. T. Botsford, 1905.
19. "Les eléments de l'art Arabe," Jules Bourgoin. Paris, 1879.
20. "Pattern Design," Lewis F. Day. London, B. T. Botsford; New York, Scribner's Sons, 1903.
21. "Geometrische Ornamentik," L. Diefenbach. Berlin, Max Spielmeyer.
22. "Romano-British Mosaic Pavements," Thomas Morgan. London, 1886.
23. "Decorated Windows, A Series of Illustrations," Edmund Sharpe. London, 1849.
24. "Specimens of Tile Pavements," Henry Shaw. London, 1858.
25. "Specimens of Geometrical Mosaics of the Middle Ages." Sir Matthew Wyatt. London, 1848.
26. "The Teaching of Geometry," David Eugene Smith. Boston, Ginn & Co., 1911.

Problems Involving Loci.

(a) Phraseology.—While the committee does not wish to prescribe the exact phraseology of any definition, it would recommend greater care in the formation of the definitions underlying the subject of loci. It is suggested that any definition used should be substantially equivalent to the following:

The locus of a point (or the locus of points) satisfying given conditions is a configuration such that:

(1) All points lying on the configuration satisfy the conditions;

(2) All points satisfying the conditions lie on the configuration.

It would seem desirable to make all proofs on loci conform to this definition. It is of course understood that the teacher will lead the pupil up to such a definition through varied forms of concrete description, such as "path of a point in motion," etc.

(b) Motion in Geometry.—It seems well to give some consideration to the place of motion in a well-rounded course in elementary geometry, and to bear in mind that this course is all the geometry to be studied by the majority of high school pupils. It has recently been urged by prominent European mathematicians that motion should be given a more prominent place at this stage. We may well recall that the space concepts dealt with in our usual courses in geometry are almost entirely to be described as static. There is in theorems and problems on loci a dynamic element that is of importance. The pupil is pretty familiar with motion as a concrete experience, and it seems of first-class importance to idealize some such concrete experiences, until they possess the precision of geometry.

For example, in a given plane we may consider in a way well described as a static configuration the perpendicular bisector of a line-segment joining two points; but when we consider this line as generated by a point moving in the plane in such a way that it is always equidistant from the two given points, we add a dynamic element.

As to phraseology, the expression "locus of points" suggests a static configuration, while the expression "locus of a point" emphasizes the dynamic element, and is equivalent in thought to the "path of a point moving with certain prescribed conditions." Both of these phases should have a place in the treatment of loci problems, and thus both forms of expression should

be used, the one or the other being more suggestive in different cases. It is even desirable to use different forms in describing a given case to make clear the idea and to cultivate facility in expression.

(c) Concrete Nature of Loci.—Contrary to the usual conception, the locus idea is one that may very easily be made concrete and brought down to the comprehension of young pupils. For example, the opening of a book or of a door suggests a variety of loci. The same may be said of many concrete illustrations easily accessible to the pupil.

In this way, loci problems may and should be introduced at certain stages of the subject. For example, in Book I: The locus of a point equidistant from two fixed points, equidistant from two intersecting lines, or from two parallel lines, or at a given distance from a fixed line. In the book on circles, the locus of all points equidistant from a fixed point, the locus of the centers of circles of fixed radius and tangent to a given line, the locus of the centers of all circles tangent to two parallel lines or two intersecting lines, and the locus of the vertices of all triangles having a common base and equal vertex angles. In solid geometry, the locus of points equidistant from a given point, from two given points, from a given plane, from two intersecting planes, from two parallel planes, etc.

(d) Loci in Problems of Construction.—Important features of the construction problems in geometry are dependent upon loci considerations which should be emphasized in this connection. For example:

(1) To find the locus in the plane of all points equidistant from three given points, it is necessary to determine the intersection of two loci both of which are straight lines.

(2) To find the locus in the plane of all points equidistant from a fixed point and at a given distance from a given line, it is necessary to find the intersection of two loci one of which is a straight line and the other a circle.

The discussion of the various possibilities in connection with such problems is one of the most valuable exercises for the pupil. For example, as to whether there are one, two, or no points fulfilling the conditions in the second example above. While it may be possible to solve and discuss such problems without

using the term locus at all, yet this leads to roundabout and awkward explanations while the language of loci is elegant and concise.

Moreover, facility in the use of this language is not only desirable from the standpoint of the high school pupil but is of the utmost importance for those who may continue the study of geometry in college.

(e) To summarize, the locus idea is deserving of a careful and systematic treatment for the following reasons:

(1) It introduces a dynamic element through the consideration of the idea of motion.

(2) It presents an elegant language for the statement of those propositions on which nearly all of our problems of construction are based.

(3) It aids greatly in the cultivation of space intuition and in emphasizing the important concept of functionality.

(f) Additional Illustrations Appropriate for Use.—

1. Find the locus of all points at a fixed distance from the sides of a triangle, always measuring from the nearest point of a side.
2. Find the locus of points such that the sum of the squares of the distances from two lines intersecting at right angles is 100.
3. Find the locus of the vertices of a regular polygon of a given number of sides that can be circumscribed about a given circle.
4. Find the locus of the midpoints of the sides of regular polygons of a given number of sides that can be inscribed in a given circle.
5. Find the locus of all points from which a given line-segment subtends a given angle.
6. Find the locus of a point the sum of the squares of whose distances from two given points is constant.
7. Find the locus of a point the difference of the squares of whose distances from two given points is constant.
8. Find the locus of all lines drawn through a given point, parallel to a given plane.
9. Find the locus of a point in space equidistant from three given points not in a straight line.

Algebraic Methods in Geometry.

The committee feels that the use of algebraic forms of expression and solution in the geometry courses may well be extended, with advantage to both algebra and geometry, and that this may be done without in any way encroaching upon

the field of analytic geometry, which belongs to a later stage of development.

(a) The Notation Should be More Algebraic.—While it is not feasible or desirable to lay down hard and fast rules to standardize the notation of geometry, an examination of current texts makes it evident that some notations in common use are unnecessarily awkward when compared with the notations used in elementary algebra. The notation of geometry is, in general, improved by much use of lower-case letters to represent numerical values, leaving capitals to represent points. This notation is here called algebraic because the student will recognize the relations of equality and inequality much more readily in the familiar notation of algebra than if these relations are presented in a notation not used in algebra.

(b) Algebraic Statement of Propositions.—Many of the theorems of geometry may be stated to advantage in algebraic form, thus giving definiteness and perspicuity and especially emphasizing the notion of functionality. This mode of expression can be made of much value to the student if he is required to translate into English all the symbols involved.

The following are illustrations of the algebraic statement of propositions:

(1) In any triangle, $a = bh/2$, where a is the area, b is the base and h is the altitude.

(2) In a right triangle, $c^2 = a^2 + b^2$ where c is the hypothenuse and a and b are the sides including the right angle.

(3) In any triangle, $c^2 = a^2 + b^2 \pm 2ap$, where a, b, c are sides of the triangle and p is the projection of b on a.

(4) For any secant and tangent drawn from a point to a circle, we have $t^2 = sx$, where t is the length of the tangent, s is the length of the secant and x is the length of the external part.

It is not intended to convey the impression that the usual statement of propositions should be replaced by the algebraic statements but rather that the student should be required to translate the one form of statement into the other. The algebraic statements are often superior to the usual statements in point of brevity and conciseness. Moreover, the algebraic statement prepares for the idea of functionality which is too little understood by persons who are not trained in mathematics beyond the high school course. That is to say, some appreciation

of the influence of changing one part of a configuration on other parts of the configuration can often be gained readily from the algebraic statement.

(c) Geometrical Construction of Formulas.—Some propositions can be proved simply and elegantly by methods involving algebra. It is somewhat usual in text-books on geometry to give a proof of the geometrical statement of such an algebraic formula as $(a+b)^2 = a^2 + b^2 + 2ab$, where a and b are the numerical measures of the line segments, but to neglect the geometrical construction of the formula. The latter seems to be the point of greatest importance. It is not additional evidence of the validity of the theorem that is sought. That is established in algebra. What is of first-rate importance is to give a geometrical picture of the formula, thus showing a certain geometrical interpretation and to have the student put the result into geometrical phraseology when a and b are line-segments.

The construction of line-segments $a+b$, $a-b$, and of areas ab, $(a+b)^2$, $(a-b)^2$, where a and b are line-segments, should come early in the course. Later, when the requisite theorems are being developed, the further elementary expressions

$$ka, \frac{a}{k}, \frac{ab}{c}, \sqrt{ab}, \sqrt{a^2+b^2}, \sqrt{a^2-b^2}, a\sqrt{k},$$

where a, b, and c are line-segments and k is a positive integer, should be constructed.

This interdependence of algebra and geometry is a matter of no small importance both historically and for subsequent mathematical work. It should be brought out by suitable exercises that the use of algebra often enables one to establish relations from which a geometrical construction can be made readily or to show the nature of a difficulty involved.

For example, to inscribe a square in a semicircle:

If x represents the side of the square and r the radius of the circle, we have at once from a right triangle that $r^2 = x^2 + x^2/4$ and hence $x = \pm\, {}^2/_5\sqrt{5}\, r$, which can be constructed from exercises given above.

(d) Geometric Exercises for Algebraic Solution.—Some exercises for algebraic solution, such as are found in many recent texts, should find a place in any course in geometry. For

example, the following is a suitable exercise after the proposition stating that $a = bh$, where a is the area, b and h are sides of a rectangle:

The area of a rectangle is 480 square inches. Each side of the rectangle is increased 1 inch and, by this change, the area is increased 45 square inches. Find the sides of the rectangle.

Similarly, after the proposition pertaining to secants and tangents to a circle, the following is suitable:

A secant line which passes through the center of a circle of radius 10 is intersected by a tangent of length 15. Find the length of the external part of the secant.

Such exercises do much to unify geometry and algebra, and may well replace some of the usual exercises.

Finally, after the theorem on the volume of a frustum of a pyramid, a problem like the following has value as an algebraic exercise, although it is in no sense a real applied problem.

A pier is built of solid concrete construction, in the form of a frustum of a pyramid with square bases. The altitude is twice an edge of the lower base and the area of the lower base is four times that of the upper base. Find the dimensions of each base if the pier contains 600 cubic feet of solid concrete.

SECTION E. SYLLABUS OF GEOMETRY.

PREFACE TO LISTS OF THEOREMS.

(1) *Lists not Exhaustive.*—The lists of theorems which follow are not to be taken as exhaustive, and it is distinctly understood that theorems may be added at the discretion of the teacher. For example, the theorem on the existence of regular polyhedra may find a place in certain courses. Some theorems are omitted only with the understanding that they may be inserted as exercises for the student; some such possible exercises are:

In any triangle, the product of any two sides is equal to the product of the segments of the third side formed by the bisector of the opposite angle, plus the square of the bisector.

The medians of a triangle meet in one point which divides each median in the ratio 1 : 2.

To divide a given straight line-segment in extreme and mean ratio.

To find the area of a triangle in terms of its sides.

To construct a square having a given ratio to a given square.

The surface of a sphere is equivalent to the area of four great circles.

(2) *Logical Order.*—Although there is some indication of a possible logical order in the lists, there is no intention of specifying any definite order. It would be impossible to carry out as a whole precisely the order stated below.

In several connections the words "corollary to" or "synonymous to" may *seem to imply an order.* These phrases are used only to indicate the reason for putting the theorem quoted in the group in which it appears. Thus in Group II., on page 71, it would not be clear in every case that each theorem is a corollary of plane geometry without such a suggestion of possible derivation.

It should be noticed that some logical arrangements would necessitate the insertion of the theorems omitted in this list. Such an insertion is entirely in the spirit of this report, as is also any conceivable change in the order, except where specified explicitly in the report.

(3) *Subsidiary Theorems.*—A number of theorems omitted in the lists below may well be given as ordinary statements in the course of the text as corollaries, or as remarks, without the emphasis which attaches to formal theorems. Among such general statements which should by all means be made at the proper points are the following:

No triangle can have more than one right angle or more than one obtuse angle.

The third angle of a triangle can be found if two are known.

An equilateral triangle is equiangular.

The square on a side of a right triangle adjacent to the right angle is equal to the square on the hypotenuse minus the square on the other side.

Through three points not in a straight line not more than one plane can be passed.

The areas of two spheres are to each other as the squares of their radii; their volumes as the cubes of their radii (like statements for other solids).

The number of such statements is exceedingly large and all of them could not be given in any syllabus. A large majority are at the present time stated, if at all, in the course of the reading matter, or in exercises, and not as explicit theorems. It is understood, and indeed expected, that these statements, together with many which are omitted from the lists of theorems below, should be treated in this manner.

(4) *Informal Proofs.*—The theorems given below under the heading: "Theorems for informal proofs," should be stated at the proper points in the text and in theorem form, or as postulates. Their proofs, however, can well be omitted where this omission is suggested, or be made exceedingly informal by the insertion of a single phrase which will give the proper suggestion for the proof. Many other theorems which are equally obvious are not stated because they occur more naturally as corollaries or as exercises. (See the preceding paragraph.)

Regarding the method of proof in general, while the demonstrations should remain as logical as they are at present, it is suggested that the *formalities* of logic, as such, be frequently dispensed with to a very considerable extent and that the propositions be frequently stated and proved in language resembling that to be found in any other mathematical text-book. This is, indeed, the style of many classical treatises, such as Legendre's or Euclid's. It is certainly satisfactory and there is no reason why the proof should not remain quite as logical when the older style is followed.

The symbolic form of demonstration which appears in many texts should be regarded simply as a shorthand expression of a complete proof in ordinary English phraseology. The latter should be given by the student in all cases. The ability to pass from the symbolic form to ordinary English, that is, to translate the shorthand into the language of everyday life, should be constantly tested by the teacher, for the same reason that the formulas of algebra derive their real meaning and power from the thought content which the student can attach to them.

(5) *Arrangement for Emphasis.*—The main list of theorems is divided into several heads, each group being introduced by a theorem of suitable importance upon which the rest of the theorems in that group depend more or less closely. This arrangement has been selected in order to emphasize the importance of a few major propositions, namely, those which carry a maximum of applications and from which the rest can be derived, thus serving as a nucleus for the whole of geometry.

This effort to gain emphasis has been carried out still further by printing the theorems in different grades of type so that those of fundamental importance and of basal character are

printed in black face type; those of considerable importance which are secondary only to the preceding ones are printed in italics. A number of other theorems are printed in Roman type, while the least important are printed in small type. The latter (small type theorems) may be omitted without serious danger, or they may be used as corollaries or exercises instead of receiving the emphasis which attaches to a theorem; in fact, probably no injury would result from a similar treatment of many of the theorems stated in Roman type.

The distinction in emphasis is desirable not only for guidance in omitting theorems in courses which are necessarily abbreviated, but it is also of the highest importance in courses in which all of the theorems are given. An orderly classification of theorems in the student's mind, a notion of the dependence of the minor theorems on the more basal ones and an appreciation of their relative importance is of the utmost direct value to the student and furnishes him with the only possible means of permanently retaining geometrical knowledge in usable form. The direct value mentioned arises both from the power acquired and also from the essential grasp of the subject, which is the purpose of education. It is a fundamental characteristic of the mind from which there is no escape that any clear impression of a vast field must have exactly such distinctions in emphasis as are outlined here for geometry. These statements and this arrangement are intended to be of assistance to the teacher in guiding him as to the emphasis to be laid upon theorems during the course and especially at the completion of a given book or chapter.

(6) *Trigonometric Ratios.*—Attention is called to the paragraphs under XIII., 2–4, on the computation of two-place tables of *sines, cosines,* and *tangents* from actual measurements, provided the pressure of time due to examining bodies is not too great. This work can be done with about the same amount of effort that is expended by the student on the ordinary geometrical theorems of the same class. Its importance is due to the fact that such a small table will really present the fundamental ideas of trigonometry and will enable the student to solve right triangles in the trigonometric sense.

(7) *Abbreviations.*—In a large number of instances theorems are stated in condensed or abbreviated form and the statement of a number of theorems is often combined into one. This is done only for the purpose of reducing the length of this report. It is to be understood that such abbreviated statements are made only for the teacher and should not be presented to the student in this form. In particular, it is probably preferable to use words instead of letters in statements for high school pupils of such theorems as those in II., 1–4.

The numbers which follow each of the theorems are references to a syllabus prepared by a committee of the Association of Mathematics Teachers of New England, 1906.

(8) *Omissions.*—Since the quantities which appear in geometry are often treated by means of their numerical measures, the introduction of any useful algebraic facts is completely justifiable; in particular, the algebraic *theory of proportion* may well be employed; but such algebraic material is not included in this syllabus since the syllabus deals primarily with geometric topics.

The following list shows the omissions from the New England Syllabus:

Plane geometry omissions: C2, G12 (2nd part), G20, J13, L2, N6 (2nd part), N13, P1, P2 (see note to XV., 1), P6, P12, T2, T4, T5.

Solid geometry omissions: E7, E8, E9b, F12, F13, F14, H2 (but see IV., 7), K2, K3 (see note to I., 2) M6, M7, M8, M9, Q3, Q4, Q8b, Q9 (see VI., note), R1, R2 (see note to VI., 13), R9, R10, R13, R14 (see note to VI., 22), S3 (see preface, 3), S7.

Theorems of Plane Geometry.

I. Theorems for Informal Proof.

(The following theorems may be stated as assumptions, or may be given such informal proof as the circumstances may demand.)

1. All straight angles are equal.* [*]

* Reference numbers are to the New England Syllabus. Where an asterisk [*] replaces the reference number, the theorem is not contained in that syllabus.

2. All right angles are equal. [*]

3. The sum of two adjacent angles whose exterior sides lie in the same straight line equals a straight angle. [J1.]

4. If the sum of two adjacent angles equals a straight angle their exterior sides form a straight line. [J2.]

5. Only one perpendicular can be erected from a given point in a given line. [G3.]

6. The length of a circle (circumference) lies between the lengths of perimeters of the inscribed and circumscribed convex polygons. [P13.]

(It is recommended that this statement be used as a definition to be inserted at context.)

7. The area of a circle lies between the areas of inscribed and circumscribed convex polygons. [P14.]

(It is recommended that this statement be used as a definition to be inserted at context.)

8. Two lines parallel to the same line are parallel to each other. [*]

9. Vertical angles are equal. [J3.]

(Very informal proof sufficient.)

10. Complements of equal angles are equal. [*]

11. Supplements of equal angles are equal. [*]

12. The bisectors of vertical angles lie in a straight line. [J4.]

13. Any side of a triangle is less than the sum of the other two and greater than their difference. [*]

14. A diameter bisects a circle. [A5.]

15. A straight line intersects a circle at most in two points. [G6.]

II. Congruence of Triangles.

1. Any two triangles* ABC and A'B'C' are congruent if:

* In this syllabus the angles of a triangle *ABC* are denoted by the capital letters *A*, *B*, and *C*; the sides are denoted by small letters *a*, *b*, and *c*, where *a* is the side opposite the angle *A*, etc.

(1)	$a = a'$	$b = b'$	$C = C'$	[A1.]
(2)	$a = a'$	$B = B'$	$C = C'$	[A2.]
(3)	$a = a'$	$b = b$	$c = c'$	[A3.]
(4)	$a = a'$	$c = c'$	$C = C' = 90°$	[A4.]

(State these in detail and in English. See preface, article 7.)

2. A triangle is determined when the following are given: (1) a, b, C; (2) a, B, C; (3) a, b, c; (4) a, c, C=90°. [*]

(Synonymous to 1.)

3. Construction of triangles from given parts; measurement of unknown parts by ruler and protractor. Given: (1) *a, b,* C; (2) *a,* B, C; (3) *a, b. c;* (4) *a, c,* C, possibly two solutions. [*]

(This is the fundamental, elementary idea of trigonometry.)

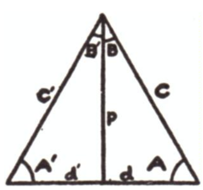
4. In any two triangles if $a = a'$ and $b = b'$, either of the inequalities $c > c'$ or $C > C'$ is a consequence of the other. [O3, O4.]

III. Congruent Right Triangles.

1. Two right triangles are congruent if, aside from the right angles, any two parts, not both angles, in the one are equal to corresponding parts of the other. [A4.]

(Very important subcase of II, 1.)

2. If two oblique lines c and c' be drawn from a point in a perpendicular p to a line AA', cutting off distances d and d', then any one of the equalities, $c = c'$, $d = d'$, A = A' B = B', is a consequence of any other. [G5.]



3. *A diameter perpendicular to a chord bisects the chord, the subtended angle at the center, and the subtended arc; conversely, a diameter which bisects a chord is perpendicular to it.* [G5b, G8.]

(Corollary to 2. See also IV, 3.)

4. If two oblique lines c and c' be drawn from a point in a perpendicular p to a line AA', cutting off unequal distances d and d', then either of the inequalities $c > c'$, $d > d'$, is a consequence of the other. [O5, O6.]

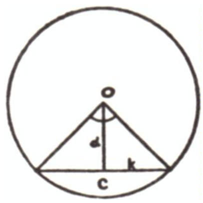
(In particular, c is greater than p.)

5. *If, in a triangle ABC, $a = b$, the perpendicular from C on c divides the triangle into two congruent triangles.* [*]

6. *In a triangle ABC, either of the equations $a = b$, $A = B$, is a consequence of the other.* [G1, G2.]

7. In a triangle ABC, either of the inequalities $a > b$, $A > B$, is a consequence of the other. [O1, O2.]

IV. Subtended Arcs, Angles and Chords.



1. **In the same circle, or in equal circles, any one of the equations $d = d'$, $k = k'$, $c = c'$, $O = O'$, is a consequence of any other one of them. [A6, 7, 8, 9, G9.]**

2. Any one of the inequalities (see figure),

$$d < d',\ O > O',\ c > c',\ k > k',$$

is a consequence of any other one of them. [O7, 8.]

3. *In any circle an angle at the center is measured by its intercepted arc.* [J8.]

(Only the commensurable case.)

4. If a circle is divided into equal arcs, the chords of these arcs form a regular polygon. [G12, last part.]

5. To construct an angle equal to a given angle. [J14.]

(Regular polygons may be constructed approximately by means of a protractor. In the same way other approximate constructions may be introduced which depend upon the protractor.)

V. Perpendicular Bisectors.

1. **The perpendicular bisector of a line-segment is the locus of points equidistant from the ends of the segment. [S1.]**

2. *To draw the perpendicular bisector of a given line-segment.* [G14.]

3. To erect a perpendicular at a given point in a line. [*]

(Corollary to 2.)

4. To drop a perpendicular from a given point to a given line. [D5.]

(Corollary to 2.)

5. To bisect a given arc or angle. [G15, 15.]

(See III, 3.)

6. To inscribe a square in a circle. [G18.]

7. *One and only one circle can be circumscribed about any triangle.* [G13.]

8. Three points determine a circle. Two circles can intersect, at most, in two points; this will happen when the distance between their centers is less than the sum of the radii and greater than the difference of the radii. [G7.]

(Corollary to 7.)

9. Given an arc of a circle, to find its center. [*]

(Corollary to 7.)

10. A circle may be circumscribed about any regular polygon. [G13, **third part.**]

11. The perpendicular bisectors of the sides of a triangle meet in a point. [T3.]

VI. Bisectors of Angles.

1. The bisector of any angle is the locus of points equidistant from the sides of the angle. [S2.]

2. *A circle can be inscribed in any triangle.* **[G13, second part.]**

(Construction to be given.)

3. A circle can be inscribed in any regular polygon. [G13, **last part.**]

4. Of the inscribed and circumscribed regular polygons of n and $2n$ sides for a given circle, to draw the remaining three polygons when one is given. [G17.]

5. The bisectors of the angles of any triangle meet in a point. [T1.] (Corollary to 2.)

VII. Parallels.

1. When two lines are cut by a transversal the alternate interior angles are equal if, and only if, those two lines are parallel. [Half of D1, 2.]

When two lines are cut by a transversal, the alternate interior angles are unequal if, and only if, the lines are not parallel.

(Synonymous to 1.)

2. When two lines are cut by a transversal the corresponding angles are equal, and the two interior angles on the same side of the transversal are supplementary if, and only if, the two lines are parallel. **[Half of D1, 2.]**

(Corollary to 1.)

3. Two lines perpendicular to the same line are parallel. [D4, G4.]

(Only one perpendicular can be let fall from a point without a line to that line. Synonymous to 3.)

4. A line perpendicular to one of two parallels is perpendicular to the other also. [D3.]

(Corollary to 1.)

5. *If two angles have their sides respectively parallel or respectively perpendicular to each other, they are either equal or supplementary.* [J7.]

6. Through a given point to draw a straight line parallel to a given straight line. [D6.]

7. *A parallelogram is divided into two congruent triangles by either diagonal.* [*]

8. In any parallelogram the opposite sides are equal, the opposite angles are equal, the diagonals bisect each other. [D7.]

(Corollary to 7.)

9. In any convex quadrilateral, if the opposite sides are equal, or if the opposite angles are equal, or if one pair of opposite sides are equal and parallel, or if the diagonals bisect each other, the figure is a parallelogram. [D8.]

VIII. Angles of a Triangle.

1. In any triangle the sum of the angles is two right angles. [J5(b).]

2. In any triangle, any exterior angle is equal to the sum of the two opposite interior angles. [J5(a).]

(Synonymous to 1.)

3. The sum of the interior angles of any polygon of n sides is $2(n-2)$ right angles. [J6.]

4. To inscribe a regular hexagon in a circle. [G19.]

To construct an angle of 60°. (Synonymous to 4.)

IX. Inscribed Angles.

1. An angle inscribed in a circle is measured by half of its intercepted arc. [J9.]

2. *Angles inscribed in the same segment are equal to each other.* [*]

3. An angle inscribed in a semicircle is a right angle. [*]

4. The two arcs intercepted by parallel secants are equal. [G11.]

5. The angle between a tangent and a chord is measured by half the intercepted arc. [J10.]

6. The angle between any two lines is measured by half the sum, or half the difference, of the two arcs which they intercept on any circle, according as their point of intersection lies inside of, or outside of, the circle. [J11, 12.]

7. *The tangent to a circle at a given point is perpendicular to the radius at that point.* [L1, 3.]

8. For a given chord, to construct a segment of a circle in which a given angle can be inscribed. [J15.]

9. To draw a tangent to a given circle through a given point. [L4.]

10. The tangents to a circle from an external point are equal. [G10.]

(Corollary to 7.)

X. Segments Made by Parallels.

1. If a series of parallel lines cut off equal segments on one transversal, they cut off equal segments on any other transversal. [D9.]

2. *The segments cut off on two transversals by a series of parallels are proportional.* [See **N10.**]

(Only the commensurable case.)

3. *A line divides two sides of a triangle proportionally, the segments of the two sides being taken in the same order, if and only if it is parallel to the third side.* **[N1, 2.]**

(Only the commensurable case.)

4. To divide a line-segment into *n* equal parts or into parts proportional to any given segments. **[N9, 10.]**

5. To find a fourth proportional to three given line-segments. **[N11.]**

XI. Similar Triangles.

1. Two triangles ABC and A′B′C′ are similar if

(1)	$A = A'$	$B = B'$	$C = C'$	**[N3.]**
or (2)	$a = ka'$	$b = kb'$	$C = C'$	**[N4.]**
or (3)	$a = ka'$	$b = kb'$	$c = kc'$	**[N5.]**

where k is a constant factor of proportionality.

(See preface, article 7.)

2. *Given a fixed point P and a circle C, the product of the two distances measured along any straight line through P, from P to the points of intersection with C, is constant. This product is also equal to the square of the tangent from P to C if P is an external point.* [N18.]

3. The bisector of any angle of a triangle divides the opposite side into segments proportional to the adjacent sides. [**Half of N6.**]

4. To construct a triangle similar to a given triangle. [*]

(Drawing triangles to scale; measurements of remaining parts to scale. Basal in trigonometry.)

XII. Similar Figures.

1. *Polygons are similar if and only if they can be decomposed into triangles which are similar and similarly placed.* **[N7, 8.]**

2. Regular polygons of the same number of sides are similar. **[N14.]**

3. The perimeters of similar polygons are proportional to any two corresponding lines of the polygons. **[N15.]**

4. **The circumferences of any two circles are proportional to their diameters, thus $c = 2\pi r$, where π is constant. [P15.]**

($\pi = 3.14\ldots$ to be computed later.)

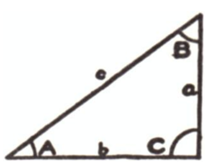
5. To construct a polygon similar to a given polygon. [*]

(Drawings to scale; maps, house plans; readings from drawings; plotting of measurements. Essential in surveying.)

XIII. Similar Right Triangles.

(The committee feels that numbers 2, 3, 4 following should have a place where time for their discussion can be secured, which will doubtless be the case except under pressure from examining bodies.)

1. **Any two right triangles are similar if an acute angle of the one is equal to an acute angle of the other, or if any two sides of one are proportional to the corresponding sides of the other. [*]**



2. *For a given acute angle A, the sides of any right triangle ABC ($C = 90°$) form fixed ratios, called the sine (a/c), the cosine (b/c), the tangent (a/b).* [*]

3. Computation of a two-place table of sines, cosines, tangents from actual measurements. [*]

(Probably a two-place table for every 5° or 10°; to be done by students, preferably on squared paper.)

4. Solution of right triangles with given parts by use of the preceding table of ratios. [*]

(Height and distance exercises.)

XIV. Right Triangles.

1. In any right triangle ABC the perpendicular let fall from the right angle upon the hypotenuse divides the triangle into two similar right triangles, each similar to the original triangle. [*]

2. The length of the perpendicular p is the mean proportional between the segments m and n of the hypotenuse; *i. e.*, $p^2 = mn$. [P8.]

3. Either side, a or b, is the mean proportional between the whole hypotenuse c and the adjacent segment m or n; that is, $a^2 = cm$; $b^2 = cn$. [P9.]

4. To find a mean proportional between two given line-segments. [N12.]

5. The sum of the squares of the two sides of a right triangle is equal to the square of the hypotenuse; $a^2 + b^2 = c^2$. [P10.]

(It should be noticed that the proposition can be proved either algebraically or geometrically.)

6. In any triangle ABC, if B is less than 90°, then $b^2 = a^2 + c^2 - 2cm$; if B is greater than 90°, then $b^2 = a^2 + c^2 + 2cm$, where m is the projection of a on c. [P17.]

(See figure under 2.)

7. Given the radius of a circle and the perimeter of an inscribed regular polygon of n sides, to find the perimeter of the regular inscribed polygon of n sides and the perimeter of the regular circumscribed polygon of $2n$ sides. [G17. **See also X4.**]

8. To calculate π approximately. [*]

XV. Areas.

1. The area of a rectangle is the product of its base and its altitude; i. e., a = bh. [P1, 2, 3.]

(This formula may be taken as the definition of area.)

2. *Parallelograms, or triangles, of equal bases and altitudes are equivalent.* [C1.]

3. The area of a parallelogram is the product of its base and its altitude; i. e., a = bh. [P4.]

4. The area of a triangle is one half the product of its base and its altitude; i. e., a = ½bh. [P5.]

5. The area of a trapezoid is one half the product of its altitude and the sum of its bases; *i. e.*, $a = \frac{1}{2}(b_1 + b_2)h$. [**P7.**]

6. The areas of similar triangles or polygons are proportional to the squares of corresponding lines. [**N16, 17.**]

7. The area of a regular polygon is one half the product of its perimeter and its apothem. [**P11.**]

8. The area of any circle is one half the product of its circumference and its radius; i. e., $a = \pi r^2$. [P16.]

9. The areas of two circles are proportional to the squares of their radii. [*]

(May be treated as suggested in preface, article 3.)

10. To construct a square equivalent to the sum of two given squares. [*]

(Pythagorean proposition.)

11. To construct a square equivalent to a given rectangle. [C3.]

(Mean proportional. See X., 6.)

Theorems of Solid Geometry.

In this part the same general principles apply as were stated in the preface above.

Throughout, but particularly in divisons I and II below, very great emphasis should be laid upon the student's real grasp of the conceptions, of the space figures, and of the significance of the theorems. While the theorems in division I will be seen to need little or no suggestion of proof, it is a mistake to suppose that they can be hastened over; on the contrary, even in these, the teacher should spare no pains to make sure that the student's mental picture is quite vivid, resorting to formal proof when necessary. To this end, illustrations, figures, models, various forms of presentation, and all such aids are legitimate throughout the course in solid geometry.

I. Theorems for Informal Proof.

1. If two planes cut each other, their intersection is a straight line. [S4.]

2. Two dihedral angles have the same ratio as their plane angles. [**K2, 3, 4.**]

(Equivalent to K3.)

3. Every section of a cone made by a plane passing through its vertex is a triangle. [**M4.**]

4. Every section of a cylinder made by a plane passing through an element is a parallelogram. [**M2.**]

5. The area of a sphere lies between the areas of circumscribed and inscribed convex polyhedrons. [*]

(It is recommended that this statement be used as a definition to be inserted at context.)

6. The volume of a sphere lies between the volumes of circumscribed and inscribed convex polyhedrons. [*]

(It is recommended that this statement be used as a definition to be inserted at context.)

7. The projection of a straight line upon a plane is a straight line. [S8.]

II. Corollaries from Plane Geometry.

(The ability to make the transfer from plane geometry to solid geometry, and vice versa, in forming conceptions and in logical deductions is of the utmost importance. The following theorems are easily reducible to plane geometry in at most two or three planes. The intention is that careful proofs be given, but the student should see that these theorems result immediately from known theorems of plane geometry.)

1. The intersections of two parallel planes with any third plane are parallel. [**F1.**]

2. A plane containing one and only one of two parallel lines is parallel to the other. [**F7.**]

3. If a straight line is parallel to a plane, the intersection of the plane with any plane drawn through the line is parallel to the line. [*]

4. Through a given point only one plane can be passed parallel to two straight lines not in the same plane. [**F10.**]

(Derived from 2.)

5. Through a given straight line only one plane can be passed parallel to any other given straight line in space, not parallel to the first. [**F11.**]

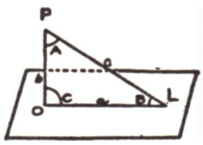
(Derived from 2.)

6. Through a given point only one plane can be drawn parallel to a given plane. [**F9.**]

(Synonymous to 4.)

7. If a perpendicular PO be let fall from a point P to a plane L, any one of the equalities

$$a=a', \ c=c', \ B=B', \ A=A'$$



is a consequence of any other of them; and any one of the inequalities

$$a>a', \ c>c', \ B<B', \ A>A'$$

is a consequence of any other one of them. **[O10, H7. See also S8.]**

8. The perpendicular PO is shorter than any oblique line. **[O9.]**

9. Two straight lines are parallel to each other if and only if they are both perpendicular to some one plane. **[F2, 3.]**

10. If two straight lines are parallel to a third, they are parallel to each other. **[F4.]**

(Derived from 9.)

11. Two planes are parallel to each other if and only if they are both perpendicular to some one straight line. **[F5, 6.]**

(Derived from 9.)

12. The locus of points equidistant from the extremities of a straight line is a plane perpendicular to that line at its middle point. **[S5.]**

13. If two straight lines are cut by three parallel planes, their corresponding segments are proportional. **[See M1.]**

14. The locus of points equidistant from two intersecting planes is the figure formed by the bisecting planes of their dihedral angles. **[S6.]**

III. Planes and Lines.

1. If a straight line is perpendicular to each of two other straight lines at their point of intersection, it is perpendicular to every line in their plane through the foot of the perpendicular. [E1.]

2. Every perpendicular that can be drawn to a straight line at a given point lies in a plane perpendicular to the line at the given point. **[E2.]**

(Corollary to 1.)

3. Through any point only one plane can be drawn perpendicular to a given line. [**E5.**]

(Corollary to 1, and II., 11.)

4. Through a given point only one perpendicular can be drawn to a given plane. [**E6.**]

(Corollary to 1.)

5. If two angles have their sides respectively parallel and lying in the same direction, they are equal, and their planes are parallel. [**K1, F8.**]

6. *If a line meets its projection on a plane, any line of the plane perpendicular to one of them at their intersection is perpendicular to the other also.* [*]

7. Between any two straight lines not in the same plane, one and only one common perpendicular can be drawn, and this common perpendicular is the shortest line that can be drawn between the two lines. [**E12.**]

8. *Two planes are perpendicular to each other if and only if a line perpendicular to one of them at a point in their intersection lies in the other.* [**E3, 4.**]

9. If a straight line is perpendicular to a plane, every plane passed through the line is perpendicular to the first plane. [**E9a.**]

(Corollary to 8.)

10. If two intersecting planes are each perpendicular to a third plane, their intersection is also perpendicular to that plane. [**E10.**]

(Corollary to 8.)

11. *Through a given straight line oblique to a plane, one and only one plane can be passed perpendicular to the given plane.* [**E11.**]

12. The acute angle which a straight line makes with its own projection on a plane is the least angle which it makes with any line of the plane. [**O13.**]

13. Two right prisms are congruent if they have congruent bases and equal altitudes. [**B1.**]

14. *If parallel planes cut all the lateral edges of a pyramid, or a prism, the sections are similar polygons; in a prism, the sections are congruent; in a pyramid, their areas are proportional to the squares of their distances from the vertex.* [**M1.**]

(See II., 13.)

15. Every section of a circular cone made by a plane parallel to its base is a circle, the center of which is the intersection of the plane with the axis. [**M5.**]

16. Parallel sections of a cylindrical surface are congruent. [**M3.**]

IV. Spheres.

1. **Every section of a sphere made by a plane is a circle. [M10.]**

(Several corollaries may be added.)

2. The intersection of two spheres is a circle whose axis is the line of centers. [**H4.**]

3. *The shortest path on a sphere between any two points on it is the minor arc of the great circle which joins them.* [**O14.**]

4. A plane is tangent to a sphere if and only if it is perpendicular to a radius at its extremity. [**M11, 12, 13.**]

5. A straight line tangent to a circle of a sphere lies in a plane tangent to the sphere at the point of contact. [*]

6. The distances of all points of a circle on a sphere from its poles are equal. [**H1.**]

7. A point on the surface of a sphere, which is at the distance of a quadrant from each of two other points, not the extremities of a diameter, is the pole of the great circle passing through these points. [**H3.**]

8. A sphere can be inscribed in or circumscribed about any given tetrahedron. [**H5.**]

9. *A spherical angle is measured by the arc of a great circle described from its vertex as a pole and included between its sides (produced if necessary).* [**K5.**]

V. Spherical Triangles and Polygons.

(Every theorem stated here may also be stated as a theorem on polyhedral angles.)

1. Each side of a spherical triangle is less than the sum of the other two sides. [**O11 (b). See also** (a).]

2. The sum of the sides of a spherical polygon is less than 360°. [**O12 (b). See also (a).**]

3. The sum of the angles of a spherical triangle is greater than 180° and less than 540°. [**K8.**]

4. *If A'B'C' is the polar triangle of ABC, then, reciprocally, ABC is the polar of A'B'C'.* [**K6.**]

5. *In two polar triangles each angle of the one is the supplement of the opposite side in the other.* [**K7.**]

6. Vertical spherical triangles are symmetrical and equivalent. [**C8, H6.**]

7. Two triangles* on the same sphere are either congruent or symmetrical if

	$a = a'$	$b = b'$	$c = c'$	[B2 (b). See also (a).]
or	$a = a'$	$b = b'$	$C = C'$	[B3 (b). See also (a).]
or	$a = a'$	$B = B'$	$C = C'$	[B4 (b). See also (a).]
or	$A = A'$	$B = B'$	$C = C'$	[B5 (b). See also (a).]

8. Either of the equations $a = b$, $A = B$ is a consequence of the other. [*]

* The same notation is used as in the plane triangles.

VI. Mensuration.

(The relation between the areas and volumes of similar solids may be treated as corollaries in individual cases. See preface, article 3. It is understood that certain statements concerning limits may be assumed either explicitly or implicitly; these are not stated as theorems. See Q3, 4, 9, R9, 10.)

1. An oblique prism is equivalent to a right prism whose base is a right section of the oblique prism and whose altitude is a lateral edge of the oblique prism. [C4.]

2. A plane passed through two diagonally opposite edges of a parallelepiped divides it into two equivalent triangular prisms. [C6.]

3. *The lateral area of a prism is the product of a lateral edge and the perimeter of a right section.* [Q1.]

(Corollary of Plane Geometry.)

4. *The lateral area of a regular pyramid is one half the product of the slant height and the perimeter of the base.* [Q2.]

(Corollary of Plane Geometry.)

5. *The lateral area of a right circular cylinder is the product of the altitude and the circumference of the base; i. e.,* $s = 2\pi rh$. [Q5.]

6. *The lateral area of a right circular cone is one half the product of the slant height and the circumference of the base; i. e.,* $s = \pi rl$. [Q6.]

7. The lateral area of a frustum of a regular pyramid is one half the product of the slant height and the sum of the perimeters of the bases. [Q7.]

8. The lateral area of a frustum of a right circular cone is one half the product of the slant height and the sum of the circumferences of the bases. [Q8 (a).]

9. The area of a zone is the product of its altitude and the circumference of a great circle; *i. e.,* $s = 2\pi rh$. [Q10.]

(Lemma for 10 below.)

10. The area of a sphere is the product of its diameter and the circumference of a great circle; i. e., $s = 4\pi r^2$. [Q11.]

11. The area of a lune is to the surface of a sphere as the angle of the lune is to 360°. [Q12.]

12. The area of a spherical triangle is to the area of the sphere as its spherical excess is to 720°. [Q13.]

13. The volume of a rectangular parallelepiped is the product of its three dimensions. [R1, 2, 3.]

(This may be taken as a definition.)

14. *The volume of any parallelepiped is the product of its base and its altitude.* [C5, R4.]

15. *The volume of any prism is the product of its base and its altitude.* [R5, 6.]

16. *The volume of any pyramid is one third the product of its base and its altitude.* [C7, R7, 8.]

17. *The volume of a circular cylinder is the product of its base and its altitude; i. e.,* $v = \pi r^2 h$. [R11.]

18. *The volume of a circular cone is one third the product of its base and its altitude; i. e.,* $v = \frac{1}{3}\pi r^2 h$. [R12.]

19. The volume of a spherical sector is one third the product of the radius and the zone which is its base; *i. e.,* $v = \frac{2}{3}\pi r^2 h$. [R15.]

20. The volume of a sphere is one third the product of its radius and its area; i. e., $v = \frac{4}{3}\pi r^3$. [R16.]

(The wording suggests a proof, but that proof is by no means prescribed. The wording is convenient, the proof may even preferably follow 21 below.)

The following two theorems, while not thought by the committee to be indispensable, offer both for student and teacher an outlook for that larger view of geometry and of mathematics as a whole which is very desirable. They forecast important principles in future mathematical courses; they are capable of the most practical direct applications; they offer a possibility of organizing and retaining the important mensuration formulæ given above.

21. *If two solids contained between the same parallel planes are such that their sections by a plane parallel to those planes are equal in area, the two solids have the same volume.* [*]

("Cavalieri's Theorem." Formal proof should not be given.)

22. The volume of any sphere, cone, cylinder, pyramid, or prism, or of any frustum of one of these solids intercepted by two parallel planes, is given by the formula $v = \frac{1}{6}h(t + 4m + b)$, where t is the area of the upper base, b that of the lower base, m that of a base midway between the two, and where h is the perpendicular distance between the two parallel planes. [See R13, 14.]

(This formula also applies to any so-called prismatoid; it is conveniently useful in practical affairs. It should not be proved for the general case, but each separate solid mentioned above, numbers 1 to 20, can be shown to conform to this rule, by a direct check.)

Maximum or Minimum Lists.

The committee feels that it would be impossible to set up a minimum list, or a maximum list, of theorems for college entrance examinations or for other broad purposes, which would meet with any widespread approval; and that the influence of such a list, if it were generally accepted, would be pernicious in leading to a total disregard of many minor facts of geometry which deserve at least passing notice.

Recognizing, however, a practical demand for some criterion on the part of the college examiner, as well as on the part of the teacher, the committee makes the following recommendations for the guidance of examiners and teachers:

(*a*) All theorems in black face type in this syllabus should be thoroughly known. The student should be able to state each of these upon a clear suggestion of its topic; and to demonstrate each of them without hesitation in accordance with some definite logical order. He should know why they are important: in particular, he should be ready to mention other theorems in geometry closely connected with them and he should know any important concrete applications of these theorems in ordinary life.

(*b*) All theorems in italics should be known to the student substantially in the form here given, but some latitude may be allowed in combining parts of these with parts of other theorems. The student should be able to prove each of these theorems on fairly short notice and he should have a reasonable idea of the importance of each of them.

(*c*) The theorems printed in ordinary Roman type should be familiar to the student when they are stated by the examiner, and the student should be able to make a proof for any one of them if allowed a reasonable interval for thought.

(*d*) The theorems printed in small type, and indeed many other facts of geometry not given in the syllabus, may be used by the examiner with the understanding that they are to be regarded in examinations as of the nature of exercises rather than as theorems with which the student is supposed already to have considerable familiarity.

(*e*) However, the committee would suggest (1) that examination questions involving the trigonometric ratios, XIII., 2, 3, 4, p. 120, be accompanied by *alternative* questions on other topics, since some schools may not find time for these applications; and (2) that no questions be given by examiners involving proofs of theorems which may properly be taken as assumptions or as definitions, such, for instance, as I., 6, 7, page 113; XV., 1, page 121; VI., 13, 21, pages 127, 128.

CONCLUSION.

It should be said that the members of the committee are not entirely agreed as to certain minor details of this report. For example, some would place among the exercises certain propositions now in small type; others would prefer to put some theorems in black faced type which are now in italics; others would prefer three types of propositions instead of four; and some would modify certain postulates and would consider as postulates or as propositions to be demonstrated certain theorems included in the list of those requiring only informal proof. The committee does not regard these minor matters of any great consequence, and therefore wishes to be considered as approving the spirit and general tenor of the report, rather than as giving individual sanction to all such details.

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